A COMPARISON OF THE PLACE VALUE UNDERSTANDING OF MONTESSORI AND NON-MONTESSORI ELEMENTARY SCHOOL STUDENTS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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ABSTRACT

Schools following the Montessori method use individual and small-group teaching methods and hands-on, concrete materials to provide a basis for deep learning of mathematical concepts. Schools with a mostly traditional approach to mathematics teaching mainly use large-group lecture methods with little use of manipulative materials.

This study investigated the understanding of place value concepts and abilities of Montessori students by comparing task responses of 93 students in grades 1-3 in a Montessori school (n=47) and in a mostly traditional comparison school (n=46). Data collection included clinical interviews with each student. The theoretical framework used in the study was taken from Zoltan Dienes, a mathematician, who believed that mathematics is learned and created by forming layers of abstract generalizations.

Interview tasks were both gathered from the literature and created by the researcher. Procedural tasks included those that asked students to count, to identify the value of digits in a number, and to use the standard addition algorithm for multidigit numbers. Conceptual tasks included those that required students to solve two-digit addition and missing addend questions with and without materials and to solve word problems involving three- and four-digit numbers. Some tasks included large numbers into the thousands because Dienes' framework calls for increasingly abstract generalizations, which for place value means larger and larger numbers.

Data analysis using Chi-squared techniques revealed statistically significant differences favoring the Montessori students on conceptual tasks when grade 1 and all grades were compared. No statistically significant differences were found on procedural tasks between schools at any grade level or overall. Although the Montessori classrooms in the study focused on conceptual development of place value ideas before students were taught procedural algorithms and rules, students in these classrooms were still able to respond to procedural tasks just as well or slightly better than students in the non-Montessori classrooms, which mostly focused on workbook activities of a procedural nature.

Recommendations for the classroom include the use of developmentally appropriate manipulative materials to expand students' ideas of place value. In addition, students must be given the time to explore these ideas. Small-group and individualized instruction are also recommended.

FOLLOW THE CHILD

Follow the child.
Is it like following the leader?
You imitate the actions of the child?
Oh, if it could only be that simple!

Follow the child.
Is it walking behind the child?
You follow her footprints?
Oh, if it could only be that simple!

Follow the child.
Know all about her development.
Observe her interests.
Why does it seem so difficult?

Follow the child.
Respect her choices.
Encourage her creativity.
Why does it seem so difficult?

Follow the child.
Model social graces.
Encourage genuine relationships.
Why does it seem so difficult?

Follow the child.
Let her reveal her needs.
Observe her soul.
Why does it seem so difficult?

Follow the child.
Trust her judgments.
Inspire trust by trusting.
Why does it seem so difficult?

Follow the child.
Be the leader.
A delicate balance in the world.
Finding the balance is always difficult.

Dance with your children.
Dance with them.
Be there not knowing.
Be there creating.

by Beth Bronsil (in McDermott, 1996, p. 136)

Dedicated to:

My husband, Bill; and daughter, Anna. You have both taught me so much and have enriched my life beyond belief. I love you.

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FIELD OF STUDY

Major Field: Education

Mathematics Education

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CHAPTER 1
RATIONALE AND THEORETICAL FRAMEWORK

The Problem

Poor mathematics achievement of U. S. students compared to those of other countries has been of concern through many generations. Educators continue to blame this lower achievement on students' lack of conceptual understanding as a result of the overemphasis on procedural learning in schools (Brownell, 1947; Bruner, 1966; Hiebert, 1986; Skemp, 1976). This is also true for place value (Hiebert & Wearne, 1992), the topic of this study. The NCTM *Standards* (1989; 1991; 2000) were created to help mathematics teachers teach so that students learn in a more conceptual way. This study focused on the Montessori method, which predates the Standards by nearly a century, but which exemplifies many of the philosophies contained in the reform efforts of today (McNamara, 1994).

Specifically, this study compared the place value understanding of Montessori and non-Montessori elementary school students. The topic of place value was chosen for study because it is one of the first topics covered in the elementary school curriculum and the particulars of its presentation to students may greatly affect how they see mathematics as a course of study and as an activity.

For the purpose of comparison, a non-Montessori school was chosen that typified a very common approach to learning that is widespread in classrooms today: whole-class direct instruction, use of textbooks and workbooks, and little use of manipulative materials past first grade. However, a caution about terminology should be given. While

Montessori schools may differ from each other in many ways, the label *Montessori* is more closely prescribed than many of the labels that could have been used for the comparison school in this study, such as *standard* or *traditional.* The label *non-Montessori* for the comparison school is not meant to convey any judgment about the school or classrooms under study. Most terms that describe schools or classrooms also convey judgments about the value, nature, or effectiveness of instruction. For this reason, the term *non-Montessori* will be used solely to indicate the comparison classroom in this study. However, since more readers may be familiar with the curriculum and methods used in the comparison school than in the Montessori school, considerable attention is given in this chapter to describing the Montessori methods. Before discussing specific aspects of the Montessori philosophy and methods as a contrast to a traditional approach, this chapter first deals with key constructs believed to be important in learning mathematics and that are found in the Montessori method: understanding, constructivism, and representations.

Understanding

Most teachers would agree that they want their students to understand the material presented in the classroom, but a problem exists in the different definitions of understanding that are used. Skemp (1976) offers the following story to illuminate this problem:

> Suppose that a teacher reminds a class that the area of a rectangle is given by A = L x B. A pupil who has been away says he does not understand, so the teacher gives him an explanation along these lines. 'The formula tells you that to get the area of a rectangle, you multiply the length by the breadth.' 'Oh, I see,' says the child, and gets on with the exercise. If we were now to say to him (in effect) 'You may think you understand, but you don't really,' he would not agree. 'Of course I do. Look: I've got all these answers right.'...And with his meaning of the word, he does understand. (p. 20)

Skemp calls this instrumental understanding, which he defines as rules and the ability to use them. He contrasts this with relational understanding that includes knowing what to do in a problem situation and why it works. For instance, while adding two three-digit numbers involving regrouping, a child could know to "carry" any sums over 10 by putting a small "1" over the next column (instrumental understanding) or the child could understand that 10 of any number value, for example the ten's place, is equivalent to one unit of the value that is 10 times the original value, in this case the hundred's place (relational understanding). Skemp claims that relational understanding is superior to instrumental understanding because it involves fewer principles of more general application and therefore is easier to recall and use.

Such a dual perspective is common in defining "understanding" (Brownell, 1947; Hiebert & Lefevre, 1986) and, in fact, some believe that different types of understanding enhance each other. For instance, Hiebert and Lefevre (1986) claim that conceptual knowledge (defined by them as a connected web of knowledge) and procedural knowledge (the use of symbols, rules, and algorithms) enhance each other: (1) Conceptual knowledge helps students develop meaning for symbols and to recall and use procedures; and (2) Procedural knowledge helps enhance concepts, apply concepts to solve problems, and promotes development of new concepts.

It is important to remember that many students, even first graders, may rely only on what Hiebert and Lefevre (1986) call procedural knowledge to solve mathematics problems. The current study is concerned with learning what types of understanding children have as well as what it is that students understand about mathematical concepts, specifically place value ideas. This dual perspective of understanding will also be used to organize results later in the data analysis sections.

Constructivism

Helping students increase their understanding of mathematics must begin with knowing the general thinking processes of children and also how they think about mathematical concepts. The constructivist theory of learning posits that all knowledge and learning are constructed by the learner (Ernest, 1996). This means that students do not just absorb what is told to them, but that they actively think and process ideas. A variety of constructivist theories exist and may be placed on a continuum: on one end are those that posit that knowledge is created entirely in the individual mind, at the other end are those that posit that knowledge is created within social interactions (Bettencourt, 1993; Wertsch, 1985).

Piaget believed that the process of learning features two mechanisms: assimilation and accommodation (von Glasersfeld, 1995). Assimilation happens when information from the internal (mind) or external world of a child makes sense to her based upon her past experiences and thoughts. The new idea is assimilated into the existing framework in her mind. Accommodation happens when a new piece of information does not fit with what the child already knows. This causes consternation (disequilibrium) within the child who must either reevaluate the new information or reconstruct existing frameworks in her mind.

While Piaget did not deny that the social environment of the child was important as a stimulus for information that is either assimilated or accommodated (von Glasersfeld, 1995), Vygotsky (1978) believed that social learning offered opportunities the child could not get anywhere else. He believed that when a child interacts with an adult or peer in solving a problem, that child could learn and do more with the guiding of the *expert*. He coined the phrase Zone of Proximal Development (ZPD) as the difference between what the child can do on his own and what the child can do in the company of an expert. Interpreting Vygotsky's writing, Wood and his colleagues (Wood & Middleton, 1975; Wood, Bruner, & Ross, 1976) used the term *scaffolding* to describe the quality of the relationship between an adult and a child in which the adult supports the child's learning, just as a scaffold supports the construction of a building. A goal of scaffolding is to keep a child within her ZPD in two ways:

> (1) by structuring the task and the surrounding environment so that the demands on the child at any given time are at an appropriately challenging level, and (2) constantly adjusting the amount of adult intervention to the child's current needs and abilities. (Berk & Winsler, 1995, p. 29)

Research on parents and preschoolers showed that effective scaffolding was characterized by a combination of structure, warmth, and responsiveness that allowed the child freedom to engage in a task with intervention only when necessary.

Maria Montessori [1870-1952], who was the first woman in Italy to become a medical doctor, left the practice of medicine to concentrate her efforts to help children become independent thinkers and actors. The Montessori method, which predates Piaget's and Vygotsky's writings, reflects the constructivist ideas already discussed. Montessori spent much of the time in her classroom observing children and kept meticulous records. She made graphs of students' classroom activity, noting times when students were at rest, in deep concentration, or restless. Montessori believed that "the rise in the level of the [activity] is related to the qualities of more advanced intellectual work; and the straightening of the line [on an activity graph] is related to qualities of internal *construction* ..." (italics in original; translated from the Italian by F. Simmonds) (Montessori, 1917/1965, p. 108). Later in the same book, Montessori wrote:

> There is in man a special attitude to external things, which forms part of his nature, and determines its character. The internal activities act as cause; they do not react and exist as the *effect* of external factors. Our attention is not arrested by all things indifferently, but by those which are congenial to our tastes. The things which are useful to our inner life are those which arouse our interest. Our internal world is created upon a selection from the external world, acquired for and in harmony with our internal activities. (p. 160) (italics in original; translated from the Italian by F. Simmonds)

Thus, Montessori theorized that as children interact with their external world, they pay attention to those stimuli that are of interest to them and ignore others. Hence, children learn by actively constructing their own "inner life." However, this learning is not constructed in a vacuum. The Montessori teacher is responsible for carefully creating an environment to support the children's development—scaffolding the children's learning (Montessori, 1917/1965).

Representations

Von Glasersfeld, a constructivist, claims the role of the teacher should be to guide the student to forming mental "re-presentations" that are conducive to learning mathematics. That is, as students gain experience in their environment, they begin to form internal frameworks or concepts that are useful to them. For instance, von Glasersfeld states that "physical materials are indeed useful, but they must be seen as providing opportunities to reflect and abstract, not as evident manifestations of the desired concepts" (1995, p. 184). In other words, adults should not assume that, though a specific material may represent to them a specific concept, these concepts will be readily apparent to children.

Bruner (1966) theorized that children could be guided to more abstract concepts by offering different modes of representation to the student. He suggested that in teaching mathematics, first a concrete representation (e.g., base-ten blocks to represent place value concepts of our number system) could be used, then iconic representations (pictures of objects), and finally abstract representations (symbols).

Lesh (1979) hypothesized that even more representations than Bruner's three are necessary. His five representations are: manipulative models, pictures, real-life scripts, spoken and written words, and symbols. Lesh claims that in order to know a concept well, students and teachers must be able to translate between representations (e.g., know that five ten-bars and four little cubes can mean 54) and transform within representations (e.g., know that 54 can be written as five ten-bars and four little cubes but also as four ten-bars and 14 little cubes). Thus, the goal of instruction for a particular topic would focus on student proficiency with different types of representations and on the ability to translate between the representations.

Dienes (1971a) also believed that children should begin to learn mathematical concepts by using physical materials and by playing games. His "theory of mathematics-learning" (p. 18) includes several principles: (1) age-appropriate, structured games should be provided to build mathematical concepts; (2) construction should always precede analysis; (3) the most general form of a mathematics concept should be included in order to provide the widest application; and (4) multiple embodiments of physical materials should be provided for each concept to promote the greatest degree of abstraction. Therefore, representations of concepts in the form of these multiple embodiments must be explored by students, and not used just as props for teachers (Dienes & Golding, 1971).

The Montessori method of teaching involves many representations. Montessori believed that "children show a great attachment to the abstract subjects when they arrive at them through manual activity" (1948/1986, p. 12). She contended that children must be allowed to develop their entire selves: emotionally, physically, as well as intellectually. This is a guide that should be used when preparing an environment for children. She observed the "great attachment" children have to physical materials that can be experienced by the child and proceeded to create an assortment of materials, pleasing to the senses and intriguing to the intellect, for use in her classroom.

Thus, Montessori believed that to help students develop abstract concepts of mathematics, a developmental approach must be used when choosing physical materials to represent mathematical concepts. However, remembering von Glasersfeld's cautions, teachers should be careful not to assume that these representations will automatically cause students to learn concepts. Of ultimate importance are the actual constructions that students create in their minds, or, in von Glasersfeld's words, the re-presentation of the concepts. These re-presentations may not match the concepts the teacher was attempting to teach. Therefore, based upon the work of Montessori and the cautions of von Glasersfeld, classrooms must provide more than physical materials to children, but also guidance in using the materials and in forming the desired concepts. Montessori agreed when she described materials as just a step in the process toward a child's desire to "reason in the abstract" (Montessori, 1917/1965, p. 83).

An Analytical Framework for Describing Classrooms

Several groups of mathematics educators, interested in how children learn mathematics and what classroom methods and environments are most conducive to this learning, have developed a framework to describe five common effective features in their own elementary school mathematics projects (Hiebert et al., 1997). These five common elements provide a framework for examining classrooms. Within these elements, a range of types of classrooms from traditional to reformed are described in the following sections. Since the Montessori classroom is generally less well known to readers than either traditional or reformed classrooms, a more detailed examination of the Montessori method is included in a separate section. Chapter 4 will return to these elements as a framework to describe the specific classrooms of the students in this study to give the reader an indication of the specific contexts of the children's learning.

1. Tasks

The nature of classroom tasks is the first element of the Hiebert et al. (1997) framework. In more traditional classrooms, tasks given to students are predominantly those involving problems in a textbook nearly identical to those modeled by the teacher in class. The emphasis of these mathematical exercises is "practice in manipulating expressions and practicing algorithms as a precursor to solving problems" (NCTM, 1989, p. 9). In Hiebert et al.'s reform classrooms, children are encouraged to invent and examine methods for solving problems.

2. Role of the Teacher

The NCTM *Professional Teaching Standards* (1991) call for a shift in teaching practices away from the more traditional view of the "teacher as the sole authority for right answers" (p. 3). To promote understanding, teachers must provide direction for students by selecting and designing tasks with mathematical goals in mind and by sharing mathematical conventions, students' methods of solution, and alternate methods. Teachers also guide the development of a classroom culture that supports students' reflection and communication, the third aspect of the framework.

3. Classroom Culture

A classroom is a community of learners and should include shared beliefs that mathematics involves collaboration, treatment of mistakes as situations for learning, and determination of correctness of a method by the logic of mathematics. This reflects a difference from the more traditional view of classrooms "as simply a collection of individuals" (NCTM, 1991, p. 3).

4. Tools

The fourth aspect involves the use of tools. Hiebert et al. (1997) include oral and written language along with physical materials in their definition of tools that are used to help students think about mathematics, to solve problems, to keep records, and to communicate mathematics. Many types of manipulative materials have been developed for classroom use, but many teachers report using them minimally, if at all (Driscoll, 1981). However, even when students do use tools, it must not be assumed that they understand mathematical concepts; students create their own meaning from the materials (Ball, 1992).

5. Equity

The last aspect of the framework deals with equity in the classroom. In a classroom that is equitable, tasks are accessible to all students and every student is heard and contributes to the discussion. More traditional classrooms, with an emphasis on direct methods of instruction, may not meet the needs of every child in the classroom, each of whom has different interests and capabilities (NCTM, 1989).

After a short vignette of a Montessori classroom, the next section examines the Montessori method in terms of each of the five areas of this framework.

Montessori Method and Philosophy

> There are forty little beings—from three to seven years old, each one intent on his own work; one is going through the exercises for the sense, one is doing an arithmetical exercise, one is handling the letters, one is drawing, one is fastening and unfastening the pieces of cloth on one of our little wooden frames, still another is dusting. Some are seated at the tables, some on rugs on the floor. There are muffled sounds of objects lightly moved about, of children tiptoeing. Once in a while comes a cry of joy only partly repressed, 'Teacher! Teacher!' an eager call, 'Look, see what I've done,' But as a rule, there is entire absorption in the work in hand.
>
> The teacher moves quietly about, goes to any child who calls her, supervising operatings [sic] in such a way that one who needs her finds her at his elbow, and whoever does not need her is not reminded of her existence...
>
> In the midst of such intense interest in work it never happens that quarrels arise over the possession of an object. If one accomplishes something especially fine, his achievement is a source of admiration and joy to others: no heart suffers from another's wealth, but the triumph of one is a delight to all. They all seem happy and satisfied to do what they can, without feeling jealous of the deeds of others. (Montessori, 1912/1964, pp. 346-347)

This passage is a scenario from one of Maria Montessori's first preschool classrooms. It is intended to help the reader visualize a Montessori classroom to better understand the information that is to follow.

The Montessori method consists of the philosophy, curriculum, and materials created by Maria Montessori beginning around 1900. The description below is based upon her original writings and may not reflect exactly what one would find in a current Montessori preschool or elementary school. The categories are those used by Hiebert et al. (1997). This framework is used to organize the presentation of the Montessori method for easy discussion, but also to show how the method is in line with reform efforts today, nearly 100 years after Montessori first began to develop her ideas and methods. Although this study is focused on the mathematics learning in Montessori classrooms, it is difficult to limit discussion to only mathematics, because the philosophy and method are a whole package and reach the entire expanse of schooling.

1. Tasks

A Montessori classroom is full of activity—and of children "doing" things. This is in stark contrast to the educational conditions that Dr. Montessori observed and abhorred in the early 1900s. She described children in these classrooms as "butterflies mounted on pins, ...fastened each to his place, the desk, spreading useless wings of barren and meaningless knowledge which they have acquired" (1912/1964, p. 14). In contrast, she created an atmosphere for her students in which they were free to explore and discover for themselves.

The tasks in a Montessori classroom are chosen by the child, sometimes after having been suggested by a teacher (Wentworth, 1999). Like Dienes, Montessori believed that games were important to children, by which she meant "a free activity, ordered to a definite end; not disorderly noise, which distracts the attention" (Montessori, 1912/1964, p. 180). By choosing their own activities, children are free to develop in their own way, on their own timetable. This is in keeping with constructivist tenets, which claim that this is the way children learn: by their own design. Hence, tasks in a Montessori classroom are designed to encourage reflection on concepts and to support future learning.

2. Role of the Teacher

For Montessori, the teacher's role is to provide an environment conducive for a child's natural physical, intellectual, and emotional growth. The teacher should use "scientific research" to observe the child and keep a record of the child's activity (we might call this clinical observation today). The teacher is not to interfere in the child's choice of activities or materials, but to begin to understand what the child is thinking when working with materials. In this way, teachers can develop their understanding for the natural progression of children's thinking.

Once a teacher has sufficiently developed this understanding of children's development, "the teacher must not limit her action to *observation*, but must proceed to *experiment*" (Montessori, 1912/1964, p. 107, italics in original). This means that, for an individual child, the teacher offers a material deemed appropriate based upon the clinical observation of the child and previous knowledge of what might be the next step in a child's conceptual development. This corresponds to Vygotsky's Zone of Proximal Development in that the teacher "expert" knows the progression of most children in a concept and can offer a material that may provide the next step in the progression. This should be done only after patiently waiting for the child to choose this material on her or his own.

3. Classroom Culture

This area is highly emphasized in many mathematics education reform projects, and as such, much of the work in these projects focuses on groups of students solving mathematical word problems and then discussing and negotiating the solutions together (i.e., Hiebert et al., 1997). Since Montessori believed all children are at different levels of development and should not be forced to do what others are doing, she rarely held large group lessons. She describes the highly respectful and nurturing culture in her classroom as developing from the children themselves as she abolished prizes and punishment (Montessori, 1912/1964). She observed that when children were free to become involved in activities they choose it brought about a "polarisation of attention..., the child began to be completely transformed, to become calmer, more intelligent, and more expansive; [he] showed extraordinary spiritual qualities, recalling the phenomena of a higher consciousness" (Montessori, 1917/1965, p. 68).

Montessori did believe that as children become older, say around six years of age, they are increasingly interested in group activities. Hence, group problem solving might be chosen by these older students.

4. Tools

The tools in a preschool or elementary Montessori classroom consist mostly of manipulative materials designed and/or adapted by Montessori. Her original materials were those she used with mentally handicapped children and she found some of these to be useful with "normal" children. She adapted these materials and created new ones after observing children working with them. Montessori stated that a "faulty or useless object could never attract the lively intellectual interest of the child. This interest, therefore, guides us in constructing in a perfect manner the means of learning" (Montessori, 1915/1995, p. 27). She believed that a material was a "success" if it held a child's attention and seemed to cause some understanding in the mind of the child. Four important aspects of the Montessori materials are important in this discussion: self-correcting features, isolation of difficulty, multiple layers of representation, and imbedded procedures.

Self-correcting features. As much as possible, Montessori built self-correction of error into her materials (Montessori, 1917, 1965). In working with a material, a child would become immediately perturbed when something happened that was not expected (e.g., using bars with various numbers of beads to sum together and getting a different answer the second time) and would want to do the task again to figure out the error. Here the child can be given the opportunity to reevaluate knowledge (accommodate) on her own, without intervention from the teacher.

Isolation of difficulty. Montessori claimed that concepts formed faster and better if the features of a task were isolated. For instance, Montessori used four sets of graduated cylinders to teach ideas of size (Wentworth, 1999). One of the sets contained cylinders whose heights increased but whose widths were the same. Another set of cylinders had widths that increased and heights that were uniform. The third and fourth set varied by both height and width (one had widths and heights that increased together and the other had widths that increased while heights decreased). A child uses the first two sets before moving on to the others in order to give time for individual concepts to be learned before the concepts are mixed. This way, each concept can be clearer for the child to understand and mistakes are more easily found and understood.

Another example of isolation of difficulty using place value materials would be the introduction to the idea of exchanging and of addition using the bead bars (see Figure 1.3 later in the text). A bar containing three beads and a bar containing five beads would have eight beads altogether and both bars could be replaced with a bar containing eight beads. This idea of exchanging is very important later in place value concepts when exchanging for 10 occurs, but can begin early for young children with smaller numbers.

This method of isolation of difficulty is in contrast to many mathematics textbooks that also break down mathematics into smaller pieces, but these textbook pieces may seem lifeless and dull to many students. The difference may lie in the mental activity of the student because of the type of understanding elicited. Many of the tasks in textbooks require students only to learn small pieces of procedural knowledge, such as "carrying" in addition problems with two-digit numbers (McNeal, 1995). This does not connect to the ideas of "exchanging" 10 units for 1 ten and 10 tens for 1 hundred which Montessori students have experience doing. The Montessori materials break down the ideas into separate tasks, but still connect the conceptual knowledge between the tasks.

<u>Multiple layers of representation</u>. Montessori found that after mastering a material, children could become bored and so she provided many kinds of materials involving the same concept. For instance, after students tire of using the base-ten blocks, which are large and bulky, other forms of materials representing concepts of place value or addition

could be used. Therefore, many layers of representation (multiple embodiments) of the same concept, leading toward the abstract, would be experienced by the child.

Imbedded procedures. Montessori designed many materials developed from procedures or algorithms (Montessori, 1917/1973). For instance, after a child understands one- and two-digit multiplication using many different manipulative materials, she is shown the use of a bead frame (abacus) to keep track of the results of multiplying very large numbers. After many translations of this task, this leads toward the standard paper-and-pencil algorithm for multiplication. Theoretically, this algorithm would then be understood at both a conceptual and a procedural level.

5. Equity

Helping all students to learn is a priority in education today. Montessori began investigating her methods with nonhandicapped children in the slums of Rome. She believed all children could learn if provided an environment where they could grow and develop naturally. Because of her ideas, a main tenet of the Montessori Method is to "follow the child." This means that teachers and other adults need to look to the child to teach us what he needs. In Montessori's words:

> All other factors...sink into insignificance beside the importance of feeding the hungry intelligence, and opening vast fields of knowledge to eager exploration. If we set about this task without any method, we shall find it absolutely impossible to accomplish. But we are already in possession of the secret by which the problem can be solved, having been initiated into it by the child himself in his earlier years. We are not unknown to him nor he to us, and we have learnt from him certain fundamental principles of psychology. One is that the child must learn by his own individual activity, being given a mental freedom to take what he needs, and not to be questioned in his choice. Our teaching must only answer the mental needs of the child, never dictate them. (1948/1986, p. 7)

Thus, in the Montessori method, each child is allowed to learn at his or her own pace. This method is seemingly enhanced by the children's unimpeded and autonomous use of many manipulative materials.

Although mathematics materials are only a part of a Montessori prepared environment, they are an important part. Using experimentation and observation, Montessori created and adapted materials based upon what she determined to be "children's aesthetic and sensory preferences in color, dimension, and texture" (Loeffler, 1992, p. 107). Because the study reported here investigated place value concepts held by Montessori students, it is important to give the reader an idea of the types of mathematics materials and tasks used in Montessori classrooms. It is impossible to describe all of the materials used by students to develop concepts of number and place value. Many Montessori teachers, in the Montessori tradition, do their own observation of students, and upon seeing a need for more support of a concept, provide additional, teacher-constructed materials for their students (Thompson, 1995). The descriptions below provide information about some of the original materials designed and modified by Montessori as well as some others that have been added later.

Kamii and Joseph (1988) state that in learning place value, "an understanding of 'tens and ones' requires the construction in one's head of two systems that function simultaneously: a system of ones and a system of tens. These systems have to be created by each child, through his or her own mental activity, from the inside" (p. 50). Montessori (1917/1973) seems to agree when she relates a classroom experience:

> Any teacher who has asked herself how in the world a child may be taught to express in numerical terms quantitative proportions perceived through the eye, has some idea of the problem that confronts us. However, our children set to work patiently counting bead by bead from 1 to 100. Then they gathered in two's and three's about the "thousand chain," as if to help one another in counting it, undaunted by the arduous undertaking. They counted one hundred; and after one hundred, what? One hundred one. And finally two hundred, two hundred one. One day they reached seven hundred. "I am tired," said the child. "I'll mark this place and come back tomorrow."
>
> "Seven hundred, seven hundred—Look," cried another child. "There are seven—seven hundreds! Yes, yes; count the chains! Seven hundred, eight hundred, nine hundred, one thousand. Signora, signora, the 'thousand chain' has ten 'hundred chains!' Look at it!" And other children, who had been working with

> the 'hundred chain,' in turn called the attention of their comrades: "Oh, look, look! The 'hundred chain' has ten ten-bead bars!"
> Thus we realized that the numerical concept of tens, hundreds, and thousands was given by presenting these chains to the child's intelligent curiosity and by respecting the spontaneous endeavors of his free activities. And since this was our experience with most of the children, one easily can see how simple a suggestion would be necessary if the deduction did not take place in the case of some exceptional child. (pp. 209-210)

The counting activity described in this vignette is one of the core mathematics tasks for children in Montessori preschool classes. Figure 1.1 illustrates the hundred chain. The individual beads are golden colored.

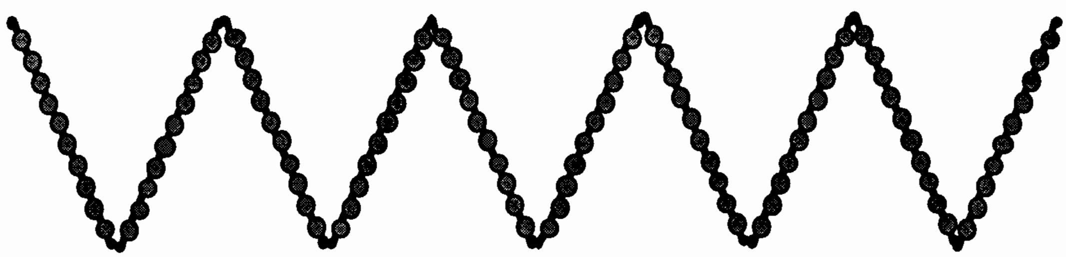


Figure 1.1: Montessori hundred chain.

As early as age 3 Montessori students are encouraged to count many objects in the classroom, including money (Montessori, 1912/1964). Children also count on a series of wooden rods, called the red and blue rods material (Figure 1.2), the shortest rod being one decimeter long and the longest one meter, which are painted alternatingly red and blue at each decimeter increment. The teacher (or directress) demonstrates counting by saying, "one" for the shortest rod, and then starting over for each rod. For example, the 4 decimeter rod would be counted by saying, "one, two, three, four."

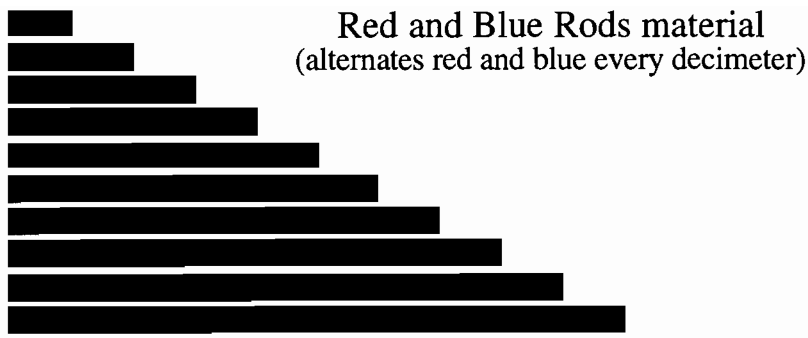


Figure 1.2: Montessori red and blue rods material.

Another counting material called a spindle box is a small wooden box containing 10 compartments, numbered from 0 through 9. Forty-five small wooden rods are placed in these compartments so that one rod is in the number 1 compartment and so on, up to nine rods in the 9 compartment. Since the sum of numerals 1 through 9 is 45, there are no rods left for the 0 compartment, which remains empty, enabling the teacher to introduce the concept of zero.

As children progress, they are shown smaller materials called bead bars (Figure 1.3) which are of different colors (Montessori, 1912/1973). For example, the 4-bar contains four yellow beads on a thin wire, while the 9-bar contains nine dark blue beads. These are used in many ways by children, including trading two bars for one of equal number: a 3-bar and a 5-bar are traded for an 8-bar and the 8-bar and a 2-bar are traded for a golden-colored 10-bar.

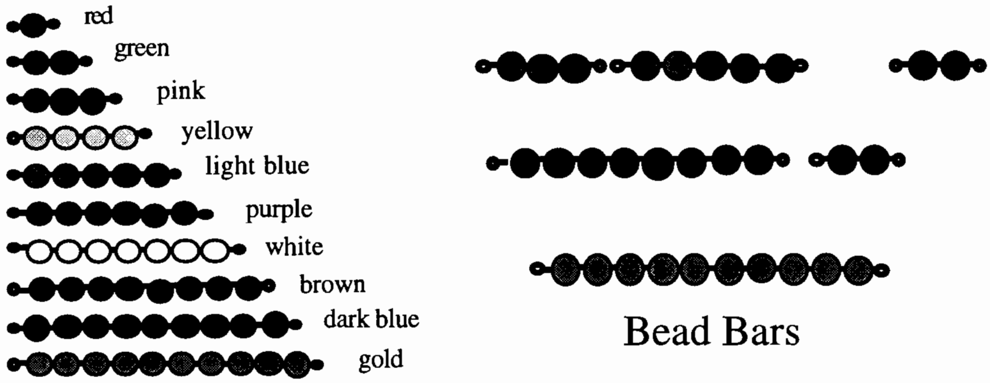


Figure 1.3: Montessori bead bars

Materials used to introduce written numerals include cards with numerals written on them (Figure 1.4). Cards with the digits 1 through 9 are the size of regular playing cards. The next set of cards contains the numbers 10, 20, ..., 90, that are twice as wide as the single digit cards. Also available are cards labeled 100, 200, ... 900, and 1000, 2000, ... 9000, which are three and four times as wide as the single-digit cards. To make the number 31, a child can place the 1 card over the 0 in the 30 card. Notice in the illustration (Figure 1.4) that the ones and one thousands place numerals are green, the tens (and ten thousands) place numerals are blue, and the hundreds (and hundred thousands) place numerals are red. This scheme follows the tradition of writing numbers using a comma between every three digits, essentially breaking up numbers into 3-digit blocks. This same color scheme is used in other Montessori mathematics materials as well.

All of these previously mentioned place-value items are available in the preschool classroom for children aged 3 to 6. Many other materials are available for children to use, including those to help with concepts of geometry, drawing, and simple fractions. In the elementary classroom, many of the same materials are used for more advanced ideas and new materials are added. For example, since the golden bead material (Figure 1.5), similar to but predating the Dienes blocks, are large and bulky, older children sometimes prefer to use the stamp game (Figure 1.6). Notice in the illustration the relative size of pieces of the golden bead material, which is not an aspect of the stamp game. However pieces in the stamp game do exhibit the green-blue-red color scheme previously mentioned.

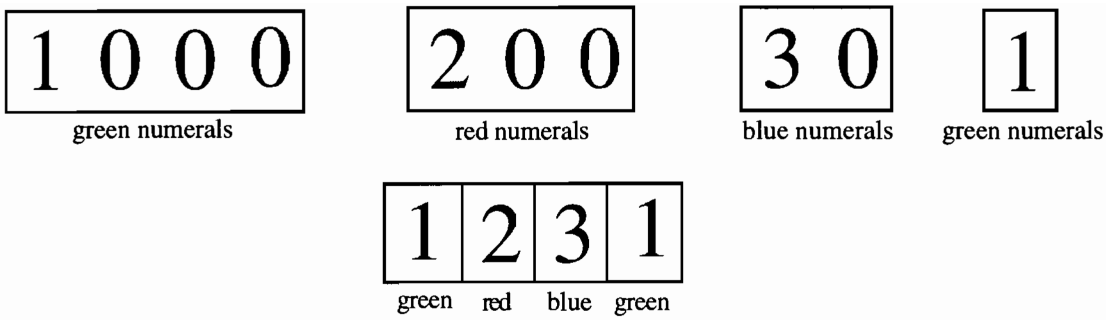


Figure 1.4: Montessori numeral cards

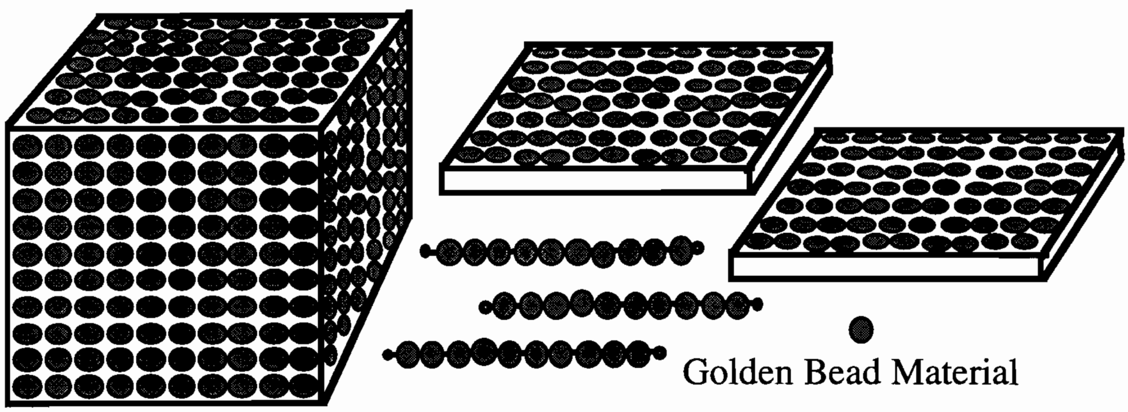


Figure 1.5: Montessori golden bead material

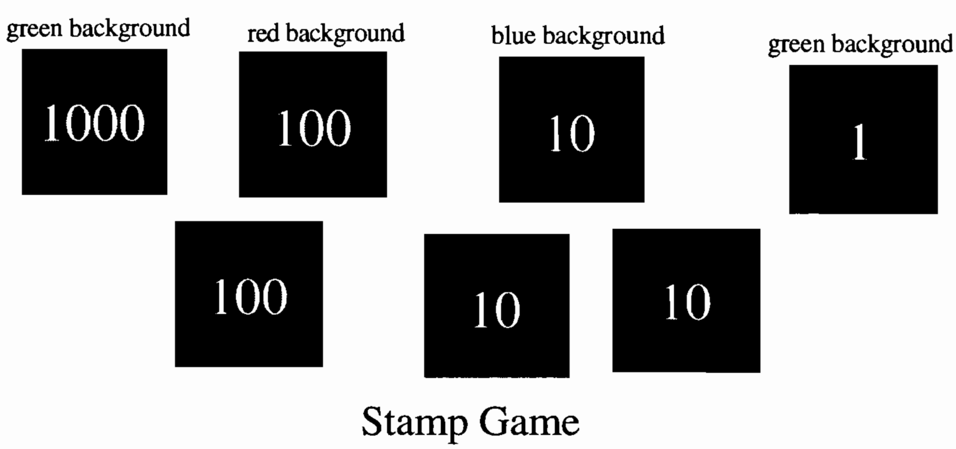


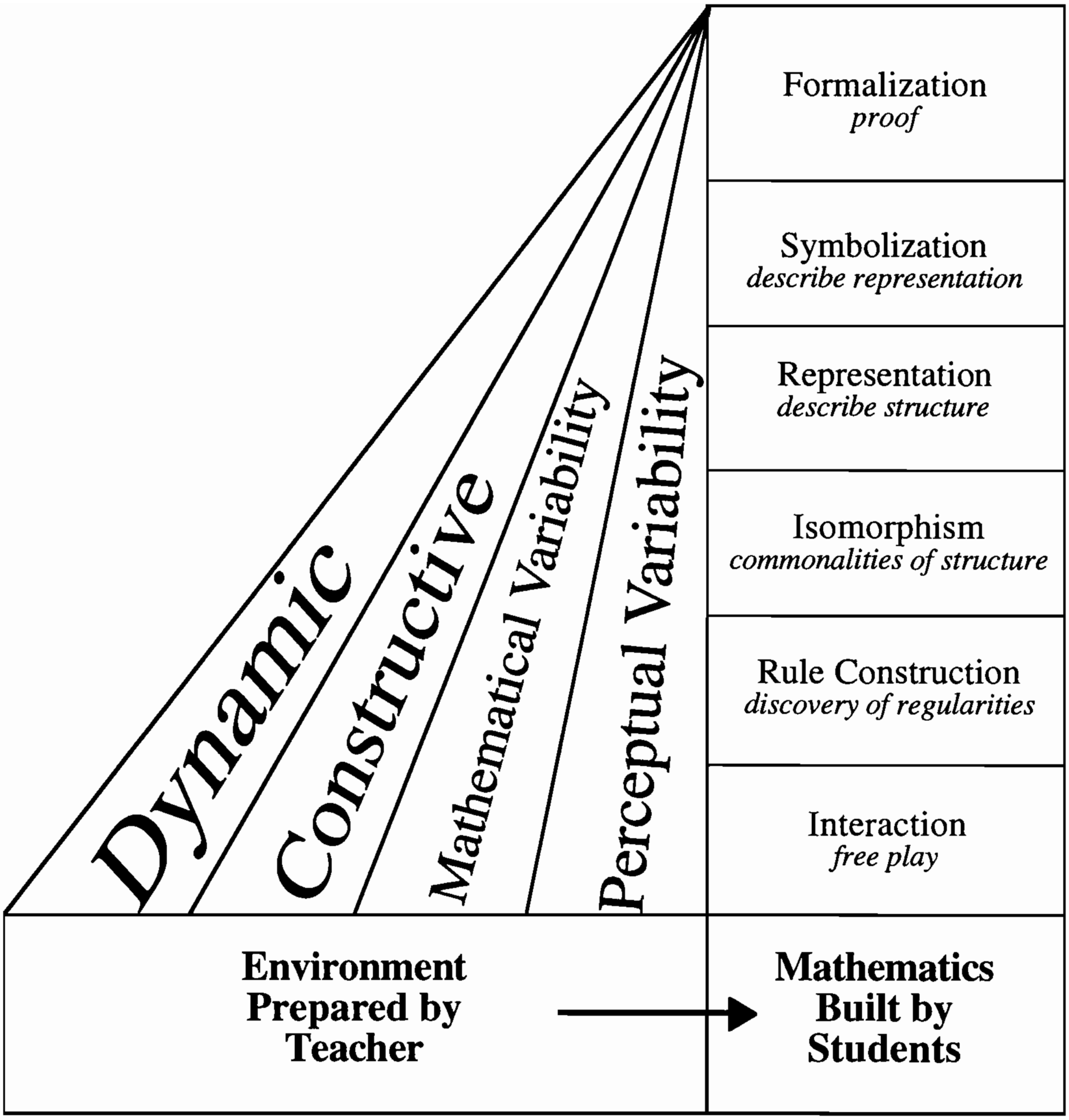
Figure 1.6: Montessori stamp game

Theoretical Framework

After having discussed the Montessori philosophy and methods, this section returns to the theories discussed at the beginning of this chapter to connect the ideas of understanding, constructivism, and representations to the Montessori system of education. This discussion lays the framework for understanding why this author believes the Montessori classroom and its students are important subjects for study.

Montessori's philosophy and methods have stood the test of time as evidenced by the large number of Montessori organizations, training centers, and schools throughout the world. However, because Montessori's ideas were developed some time ago and because they are methods for the general education of children, it is important to view these practices in light of current theory and research, specifically in the field of mathematics education.

In his book "Building Up Mathematics," Dienes (1971a), a mathematician, has proposed a theory of mathematics learning that nicely parallels the Montessori method. In this theory, mathematics "consists of structures and the relationships between structures" (Bart, 1970, p. 356). Dienes views the search for patterns to be the very essence of mathematical thinking: "Established patterns soon come to be regarded as mathematical objects, which are then fitted into further patterns; these in turn, upon becoming familiar, are regarded as objects, and so on" (1971a, p. 19). His theory describes both the gradual building of mathematical thinking and ideas and the types of activities that are conducive for students to do this "building" toward mathematical abstraction. Figure 1.7 is the author's interpretation of Dienes' theory. The tall column of mathematics processes is intended to represent the building, both as a physical and mental structure, of mathematics. The environment prepared by the teacher is meant to be pictured as a scaffold, in a physical and mental sense, alongside the building of the mathematics.



"Building Up Mathematics"

Figure 1.7: Author's interpretation of Dienes' philosophy

The discussion in this section describes in some detail Dienes' stages as applied to the ideas of place value. This description is provided to give the reader an idea of place value as a developmental process of building on previous experiences and ideas. The components of this process can be taken as the definition of place value ideas.

Interaction, the first stage of building mathematics, "corresponds [to] a rather undirected activity, seemingly purposeless—the kind of activity that is performed and enjoyed for its own sake" (Dienes, 1971a, p. 26). As a child begins to "detect regularities in his environment, he sees that the world around him can be interpreted, events predicted, by means of rules" (p. 32) in the **Rule Construction** stage. In the case of place value ideas, students in the interaction stage can do counting activities, such as using the Montessori 100 and 1,000 bead chains. Other counting activities require students to assemble the bead bars into different groupings that add to ten or other numbers. For example, in Figure 1.3, the 3-bar and 5-bar are exchanged for an 8-bar, which is in turn exchanged with the 2-bar for a 10-bar. This work can help students begin to develop the idea of grouping and exchanging, especially for the number 10, a vital step in learning place value concepts.

Looking further for more general structures, by finding similar rules in several different activities, leads to **Isomorphisms**, the third stage. Place value ideas need to be extended beyond just the idea of exchanging for 10. The notions of exchanging 10 tens for 1 hundred, 10 hundreds for 1 thousand, and so on, are vital for the ability to understand the arithmetic involving multidigit numbers. Students may be able to generalize the action of "carrying" the 1 to the next column in the standard addition algorithm, but may not understand the concept of exchanging 10 of one unit for 1 unit of the next power of 10.

Representation of the "isomorphic situations in one all-embracing, usually graphical, form" (Dienes, 1971b, p. 337) is the fourth stage of building mathematics. The use of many different forms of place value materials and the ability to understand each of them and the common features—including exchanging 10 of a unit for 1 of the next—is important for students to be able to apply the concepts and justify their solutions.

Describing the representation using symbols is called the **Symbolization** stage. In this stage, place value concepts need to be understood in order to solve arithmetic operations involving symbols and numbers. The memorization of the standard algorithms of arithmetic could be mistaken for an understanding of the ideas of place value.

Finally, **Formalization** occurs when "some properties described are chosen as fundamental...then rules are found according to which we can 'reach' the other properties from the fundamental ones" (1971a, p. 36). At the level of elementary school children, formalization can be said to be the justification of their solutions and answers. The ability to describe the process of their mathematical thinking and what rules or ideas these are based upon is key to showing a clear understanding of the underlying concepts.

These stages, which build toward mathematical abstraction, do not happen without help from a teacher. Dienes believes that some "children ... are able to gather a degree of abstraction from very scanty experience" (1971a, p. 37), but most cannot. The classroom should contain many materials or games for children's free experimentation. Also, instruction is most effective individually or in small groups:

> To allow for children's individual differences most of the learning should take place individually or in small groups of twos or threes. It is not likely that more than three children will work at the same pace and in roughly the same way. This means that all the information cannot come from the teacher, as he simply would not have time to go round and 'teach' perhaps forty different children separately, all possibly at different stages. There must be other sources of information in the classroom, as well as places to find out what to do next and, if possible, places to check the accuracy or correctness of an answer, if this is appropriate. (1971a, p. 38)

Dienes continues by suggesting that a teacher create task cards that can be chosen by students. Although the instructions on these cards are probably more prescriptive than Montessori would have agreed with, these suggestions are very much in line with her view of the classroom as a whole.

In addition to these general recommendations, Dienes proposed four principles (described below) that teachers should keep in mind when designing tasks for children. Through a carefully prepared environment (following the principles) a teacher can help students build mathematical concepts and learn to think mathematically.

Dynamic Principle

According to Dienes, to foster mathematical understanding, classrooms should be dynamic. Structured or reflective games should be provided so children can build mathematical concepts. These games are to be appropriate for the child's age and developmental level. In a Montessori classroom, children are free to use any of the materials on the shelves and are free to work with them in any way they like. Montessori did not use the term *game*, but called the child's activity *work*. Teachers observe the children and take note of what materials the children choose. The teacher can then offer appropriate materials based upon what she thinks will help the child to further his/her understanding. Of course, the child is not limited to the material suggested by the teacher. In this way, a Montessori classroom is dynamic: children are always working with materials, not just observing a demonstration by the teacher.

Constructivity Principle

For Dienes, children should always construct their own understanding of mathematics before proceeding to analysis of mathematics. The use of concrete materials that the child can manipulate individually is important for this. Later, mental games or work can be added to make the process more abstract. In a Montessori classroom, children are free to explore and construct knowledge. They are not introduced to the symbols of the mathematics until the teacher feels they are ready; after they have had many different kinds of concrete experiences (Montessori teacher interview, 9/9/99).

Mathematical Variability Principle

The largest possible number of variables should be included in any mathematical materials or games. An example of this in a Montessori classroom might be the introduction of addition and subtraction with multidigit numbers. Montessori students have had many addition and subtraction experiences with single digit numbers by making exchanges with the bead material, and then students continue with the exchanging necessary to carry out multidigit addition and subtraction. However, where most elementary curricula introduce the idea of exchanges only with two-digit numbers, Montessori students may add or subtract up to seven- or eight-digit numbers. This way, they begin to develop an understanding of the larger concept (exchanging) instead of just focusing on memorizing a procedure for adding two-digit numbers.

Another example involves the set of bead bars pictured previously (Figure 1.3). In the complete set, each number is represented by several different configurations. For example, the yellow 4-bar set includes one 4-bar, a short chain of 16 beads in the form of four 4-bars, a long chain of 64 beads in the form of four short chains of four 4-bars, a 4x4 square, and a 4x4x4 cube, all composed of yellow beads (see Appendix A). Using these materials, students are able to see squares and cubes, and can create the squares and cubes themselves. The important point here is that both squares and cubes are provided and that these squares and cubes are made for each of the numbers from 1 through 10, thus helping students to discover regularities and structures of squares and cubes, multiplication and exponent ideas, and different base systems.

Perceptual Variability Principle (or Multiple Embodiments)

Dienes believes that many types of physical materials or games should be provided when teaching a specific concept. Different children may prefer or learn better from different types of materials, and different materials involving the same concept are necessary for children to begin to see patterns across the different representations.

Montessori also provided different materials to take advantage of children's interests. As previously mentioned, the golden beads and the stamp game both involve concepts of place value, and both materials are used in the same way, but the physical embodiments are different.

Dienes theorizes that the four principles outlined here are necessary to create environments that support the learning and abstraction of mathematical concepts. As has been shown, Dienes' theory of mathematics learning parallels the methods and materials created by Montessori. If in fact the Montessori curriculum is an example of Dienes' theory "in action," a study of Montessori classrooms may provide information regarding the level of completeness or accuracy of the theory.

Statement of the Problem

The Montessori teacher and curriculum provide the freedom, opportunity, and materials for children to explore their intellectual, physical, and emotional growth. The mathematical materials, which seem to embody many of the features of mathematical concepts and procedures that are deemed important in understanding mathematics, provide representations that may help a child to construct these concepts and procedures for themselves. The role of the teacher in a Montessori classroom (to be a keen observer and researcher) provides support and guidance for children's developing mathematical understandings and constructions. Therefore, the author believes that with the rich environment provided for them, Montessori students may develop a self-motivation for learning and a rich understanding of our number system and other mathematical concepts.

This study was an exploration to show whether or not the principles put into place in the classroom by Montessori foster mathematical thinking and understanding. In order to investigate Montessori's methods, two schools were chosen to make comparisons between Montessori's methods and a more traditional view of teaching: a Montessori school and a non-Montessori school. Children in these two schools in the first, second, and third grades were interviewed using tasks involving place value concepts. Students' responses were analyzed with regard to their level in Dienes' theory of the building of mathematical thinking. The following questions guided the research:

What are the differences or similarities in the way that Montessori students complete tasks involving place value concepts as compared to non-Montessori students in first through third grade?

(a) *What are the differences or similarities in the way that Montessori students complete tasks involving counting as compared to non-Montessori students in first through third grade?*

(b) *What are the differences or similarities in the way that Montessori students complete tasks involving mental addition and subtraction as compared to non-Montessori students in first through third grade?*

(c) *What are the differences or similarities in the way that Montessori students complete tasks involving the forming and unpacking of hundreds and thousands as compared to non-Montessori students in first through third grade?*

(d) *What are the differences or similarities in the way that Montessori students complete symbolic tasks as compared to non-Montessori students in first through third grade?*

CHAPTER 2
LITERATURE REVIEW

In order to place this study within the context of other research and theory, the literature review concentrates on research conducted on the ideas of students' place value understanding and research conducted in Montessori schools. Unfortunately, no studies on the place value understanding of Montessori students were found.

Place Value Research

Number sense is one of the most valuable understandings elementary school children need to develop. Without it many children may memorize algorithms and manipulate symbols, but will not have the base of understanding about our number system that they will need to be successful in mathematics. The research on place value shows that, although they may be able to "do" arithmetic, most U. S. children below third grade do not have a good understanding of place value (Fuson, 1990).

What Is Known About Children's Place Value Understanding

After interviewing hundreds of children between first and third grade, Kamii and Joseph (1988) found that when they point to the 1 in 16, most of the children, when asked "what does *this* part...mean?," will say it means 1 and not 10. Other researchers find similar results. Even when a child can name the 1 in 16 as the ten's place, it does not mean the child understands the 1 as ten ones, but may think of place value more in terms of positional order than in terms of groupings (Bednarz & Janvier, 1982). In fact, because of the way most textbooks introduce place value and arithmetic operations requiring regrouping, children do not need to understand place value as groupings of tens because many of them can succeed by memorizing the standard computation algorithms (Cobb & Wheatley, 1988). Place value meanings are still being developed even as late as fifth grade for some children (Ross, 1989).

Bednarz and Janvier (1982) interviewed 160 children between the ages of 6 and 10 in order to better understand children's conception of number. They found that the children could compare numbers by size of those numbers, but that a strategy of digit-by-digit comparison or construction was used. More specifically, when asked to construct a three-digit number to be larger than a given three-digit number, most children constructed the number to be bigger in all places, even if that was unnecessary. Only 10% of the third graders and 30% of the fourth graders used a conception of place value to complete this task. These researchers also found that in subtraction requiring regrouping, the task of borrowing was not linked to the idea of regrouping or exchanging.

For some students a misunderstanding of place value can hinder their performance on arithmetical tasks. Interviews with 35 third-grade children showed that many were confused by the standard borrowing algorithm for two-digit subtraction, and that a misunderstanding of place value in this context was responsible for the confusion and errors (Dominick, 1991).

Many teachers may believe their students have a deeper understanding than they actually do because of the students' use of place value terminology. Even Ross (1986, April) found ambiguous results when some of her interview questions were not sensitive enough to "distinguish between the qualitatively different modes of thought" (p. 14) of the subjects. In particular, when students were asked to look at the number 52 and its representation in base ten blocks (five purple "longs" and two white blocks), they were asked if the numeral 5 had anything to do with how many blocks were there. Since the child could obviously see five purple items, many answered in accordance, although it may not have been from an understanding of place value concepts. In a follow-up study, this task was followed up with a parallel question. The children were asked to place 26 chips into groups of four. The resulting picture would show six groupings of four and two single chips left over. When asked if the numeral 2 had anything to do with the picture, many children claimed that the 2 stood for the two single chips. This illustrates the need for interview tasks that are aligned with the conceptual thinking of students, but also those that will not allow the researcher to mistakenly believe that a student knows more than he or she actually knows just because of the use of sophisticated terminology.

In another study (Cauley, 1988) focusing on regrouping in multidigit subtraction, 42 second graders and 48 third graders were placed into three groups: Group A (those students with no grasp of regrouping); Group B (partial grasp of regrouping); and Group C (proficient with regrouping). These students were then compared based on their results from the mathematics subtest of the California Test of Basic Skills. It was found that only 5% of the Group A students performed at or above one standard deviation above the mean, while 35% of Group B students and 94% of Group C students performed at or above the same mark. This suggested a link between place value understanding and arithmetical proficiency. Although these children's performance on multidigit subtraction problems and their place value understanding was shown to be positively correlated, not all students proficient in subtraction requiring regrouping necessarily had a good understanding of place value. Cauley found that of the 90 students interviewed, only 12 knew that the minuend did not change its value after regrouping.

Even when teacher candidates were interviewed about subtraction and multiplication (Ball, 1988), most could relate place value concepts to borrowing in subtraction, but many did not do the same for multiplication. Although some of the subjects used place value terminology, they did not necessarily understand this concept. Teachers cannot teach well what they do not fully understand.

Language alignment may be one cause for this lack of place value understanding and has been investigated by many researchers. Interviews with and observations of children from kindergarten through second grade were carried out in several different countries, and Asian children were found to have a better conception of number than U. S. children. The structure of the Chinese (Yang & Cobb, 1995), Japanese (Miura & Okamoto, 1989), and Korean (Fuson & Kwon, 1992; Miura, Kim, Chang, & Okamoto, 1998) languages was found to be one possible reason for this. These languages incorporate place value into their number naming systems, unlike other languages such as English, French, and Swedish (Miura, Okamoto, Kim, Chang, Steere, & Fayol, 1994). For instance, 11 is translated as "ten one," 13 as "ten three," 20 as "two ten," and 29 as "two ten nine." According to Miura et al. (1988), a result of this different number naming system is a more intuitive understanding gained by children whose first language is an Asian one. In fact, when these researchers asked Korean and U. S. kindergarten and first-grade children to show two-digit numbers, all the Korean children were able to use base-ten blocks to show a representation of the number, either at their first attempt or after being prompted to show another way if they had first used single blocks to count out the specified number. The U. S. children almost exclusively counted out the number in single blocks, even when prompted for an alternate method.

Some researchers have listed prerequisite skills or stages of development through which children must progress on their way toward a good understanding of place value. Gotow (1985) found that the level of place value understanding of 20 second graders and 28 third graders was no higher than their understanding of four prerequisite principles: "(1) synthesis of ordinal and cardinal properties of the numeration system, (2) both the addition and subtraction operations, (3) understanding of counting by groups, and (4) understanding of exchange equivalences such as one ten for ten ones" (p. 3278).

Components of Place Value Understanding

In an analysis of place value concepts, Jones and Thornton (Jones, Thornton, & Putt, 1994; Jones, Thornton, Putt, Hill, Mogill, Rich, & Van Zoest, 1996) developed a framework for teaching and assessing what they call multidigit number sense, a term used to capture the multifaceted nature of place value understanding. This framework emerged from the study of a learning program for first- and second graders, and evolved into a model containing developmental levels of four key constructions. The four constructs are: counting, partitioning, grouping, and number relationships.

Counting. Counting has been shown to be the cornerstone of learning place value concepts (Fuson, 1990). As children learn to count, they begin to construct more sophisticated ideas of number, similar to that described below:

> At the root of all quantification and of all numerical thinking and operating lies the construction of discrete, repeatable units and their conjunction. No one has formulated that basic insight more elegantly and persuasively than Joannes Carmuel who...tells this story:
>
> "There was a man who talked in his sleep. When the clock struck the fourth hour, he said: 'one, one, one, one - that clock must be mad, it has struck one four times.' The man clearly had counted four times one stroke, not four strokes. He had in mind not a four, but a one taken four times. Which goes to show that to count and to consider several things contemporaneously are different activities. If I had four clocks in my library, and all four were to strike one at the same time, I should not say that they struck four, but that they struck one four times. This difference is not inherent in the things independent of the operations of the mind. On the contrary, it depends on the mind of him who counts. The intellect, therefore, does not find numbers but makes them; it considers different things, each distinct in itself, and intentionally unites them in thought." (Joannes Carmuel, 1670/1977, pp. 43-44, translated from the Italian by Ernst von Glasersfeld in Steffe, von Glasersfeld, Richards, & Cobb, 1983, pp. 4-5)

So as children develop their ideas of number, they must construct a hierarchical system of ones as illustrated in the previous story and in Figure 2.1 (Kamii, 1986) where the number 4 is simultaneously 4 ones and 1 four.

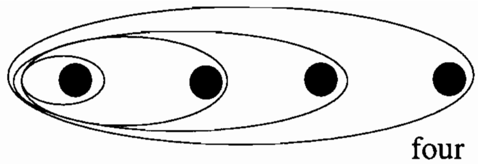


Figure 2.1: Hierarchical system of ones

Children who cannot yet simultaneously understand the individual items and the composite (in the previous case, the number 4 as 4 ones and also as 1 four) can only see the number 10 as 10 ones or as 1 ten, but not both. Steffe, Cobb, and von Glasersfeld (1988) call this **ten as a numerical composite**. When children progress and construct a number system that consists of a system of tens imposed on top of a system of ones, or in other words, when they can simultaneously "see" 10 as 10 ones and 1 ten, then they can be said to be operating with **ten as an abstract composite unit** (Cobb & Wheatley, 1988). During this phase, children can count by tens and ones.

The next advance in children's counting is to be able to count by ones and by tens in the absence of physical materials. Steffe et al. (1988) call this **ten as an iterable unit**.

Partitioning. To have a flexible understanding of multidigit numbers, children must also know how to show equivalent representations of numbers (Thompson, 1990), such as knowing that the number 34 can be broken down into 3 tens and 4 ones, but that 34 is also made of 2 tens and 14 ones. Miura, Chungsoon, Chi-Mei, and Yukari (1988) have identified three kinds of partitioning:

- one-to-one (showing 34 only as 34 single blocks);
- standard base-10 (showing 34 as 3 tens and 4 ones); and
- nonstandard base-10 (showing 34 as 2 tens and 14 ones or as 1 ten and 24 ones).

Resnick (1983) believes that children progress through these three levels as they gain understanding about multidigit numbers and that, initially, the use of physical materials is required to construct these ideas. Jones, Thornton, and Putt (1994) extend these ideas to include tasks in their framework for which students must find the missing part of a number.

Grouping. The ability to collect items into groups (e.g., by fives or tens) and to see the usefulness of such activity is essential to understanding place value concepts (Bednarz & Janvier, 1982). In fact, to understand much of arithmetic, children must understand what it means to trade 10 ones for 1 ten, an essential grouping activity.

Research comparing students speaking different languages has shown that some students (e.g., English- and French-speaking) do not naturally group objects by tens, whereas other students (e.g., Japanese- and Chinese-speaking) automatically do this (Miura, Okamoto, Kim, Chang, Steere, & Fayol, 1994). This difference may partially account for the seemingly apparent superiority of mathematics skills of some Asian students.

Number relationships. Children first learn a linear order for numbers—that numbers are in a line and increase in magnitude (Greeno, 1991). Later children develop the ability to compare numbers by their size by comparing their digits (Sowder, 1988). Sowder found that by fourth grade, students could successfully compare two 4-digit numbers or could tell which of two 4-digit numbers was closer to a third 4-digit number. These number relationship skills develop over time (Greeno, 1991) and cannot be explicitly taught to students (Sowder, 1988).

Treatments to Increase Children's Place Value Understanding

There have been many attempts to change traditional approaches to teaching number and place value in the classroom. Below are examples of research that have been shown to improve children's understanding of these topics.

Hiebert and Wearne (1992) compared conceptual-based instruction, using class discussion and manipulatives, and traditional instruction, which emphasized increased drill and practice. The conceptual group scored significantly better in place value and two-digit addition and subtraction with regrouping, even though this had not specifically been taught to either group. They also used more strategies based on an understanding of place value.

In the study just mentioned, regrouping techniques were not specifically taught to students. Engelhardt and Usnick (1991) wondered when these regrouping techniques should be introduced. They used a volunteer teacher and the 24 students in her second-grade classroom for two studies. For the first study the students were randomly assigned to two groups, both of which used manipulatives. The experimental group was taught addition activities with regrouping, using the standard addition algorithm, before they did nonregrouping addition. The control group learned addition in the traditional order: nonregrouping and then regrouping activities for addition. The results of the posttest were not significantly different for this first study. The researchers intended to continue with this same format for subtraction, but the teacher enjoyed the alternate method so much that she refused to teach the traditional method for subtraction. Therefore the second study was altered in the following way: Since the teaching of subtraction occurred later in the same year, some of the original students were no longer in the class and some new students had joined the class. Three groups of students were tested on procedural knowledge after the unit on subtraction, involving regrouping and then nonregrouping, had been taught: group A, students who had been in the previous experimental group doing regrouping first; group B, the former control group students; and group C, the students new to the class. The results were surprising. The mean scores on a posttest of the subtraction algorithm were: group A, 92%; group B, 73%; and group C, 64%. The researchers admit that the sample was very small, but the results show an interesting and exciting trend toward the success of using "instructional sequences which begin with the more general cases and examples rather than the simpler, specific cases" (Engelhardt & Usnick, 1991, p. 9).

Another study that investigated the use of a different teaching approach was attempted by Fischer (1990), who aided a kindergarten teacher in teaching four of her classes. Two classes were taught to count using the traditional count-say-write curriculum and the other two classes were taught using a part-part-whole curriculum that stressed the set-subset relationship of numbers. A typical task for these students involved having them count out a certain number of chips and then separate the chips into two piles, count the number in each pile, and relate both of those totals with the original number. A total of 82 students was involved in the study. The results showed that the students in the experimental group outperformed those in the control group on number concepts, problem solving, and place value tests. The part-part-whole curriculum seemed to be "successful in aiding [the] development of basic number concepts" (Fischer, 1990, p. 205) of the students.

Other studies have focused on the effects of introducing different ways of counting. In a study of 60 kindergartners, Hong (1989) used a counting by tens approach in which objects were grouped only by tens during counting activities. Students who used this approach maintained their skills even six months after the treatment.

Another study focusing on counting used 213 kindergartners in 11 classes (Barr, 1978). The students were put into three groups and for two months were taught using the following methods. Group A students counted only by ones even for numbers greater than 10 and, at the end of the two months, were taught place value and grouping. This place value teaching was done by using counters, for example 35 counters were placed into three groups of ten and five groups of one. Group B students never counted by ones for numbers larger than 10, grouped objects directly into tens and ones, and later learned to write numerals. The Group C students' experience was similar to those in Group A, except when the grouping training was given. At this time, the Group C students would count using tens and ones. For example, 34 would be counted as 10, 20, 30, 31, 32, 33, 34. Group A students never counted by tens, only by ones. Results on a retention test showed that Group C students outperformed Group A students, who in turn outperformed the Group B students. This was a surprising result considering the early emphasis in Group B on grouping by tens. The researchers explain the result by mentioning that when students begin school, they probably already know how to count past 10, and "it would seem that the prior experience children have in counting and reading number names before the formal introduction of 2-digit numerals makes it difficult for them to perceive this new approach (Treatment B) as something different" (Barr, 1978, pp. 41-42).

Fuson and Briars (1990) also used an altered teaching approach in order to increase the place value understanding of first and second graders. The researchers trained classroom teachers to use manipulatives and a learning and teaching approach in order to help the students link and give meaning to number symbols and their English word counterparts. Multidigit addition and subtraction were taught using base-ten blocks with a continual linking of the number symbols, English words, words for the blocks, and words for the digit cards, and subtraction was taught using a trade-first algorithm. In the first of two studies, which involved 169 students, "six of the eight classes...demonstrated meaningful multidigit addition and place-value concepts up to at least four-digit numbers" (Fuson & Briars, 1990, p. 180). A similar result was found for most second graders in a second study of 783 students in 42 schools, and "many children in the 35 classes (N=707) completing subtraction work learned at least four-digit subtraction" (Fuson & Briars, 1990,

p. 180). This shows that children can learn much more than they usually do if they are taught number relationships with conceptual understanding.

This research on place value instruction shows gains in children's understanding when using manipulative materials and a curriculum that stresses the connections between the idea of grouping by tens and the symbolic number system. The research discussed in these sections has all occurred within the last 30 years. An example of an alternative method for teaching number concepts can be found in the curriculum developed by Maria Montessori nearly 100 years ago. The next sections include findings pertaining to the Montessori method.

Montessori Research

The first investigation of the Montessori method of teaching was done by Maria Montessori herself when she opened her first *Casa dei Bambini* (Children's House) in 1907. She created her methods in the succeeding years by using the classroom as a "field for scientific experimental pedagogy and child psychology" (Montessori, 1912/1964, p. 72) and child development. Montessori believed that one of the most important qualities of a good teacher is the capacity for observation. She herself kept careful records of the types of activities and materials that were intriguing to children and also made notes about children's work habits.

Montessori Preschool Studies

The next body of Montessori research that could be located was the result of research programs designed to evaluate the effectiveness of the preschool Head Start program that began in 1964. These research programs, some of which were longitudinal experimental studies, were an attempt to find out which programs or aspects of programs would help disadvantaged children overcome their situations and become prepared for success in elementary school. Programs were chosen for study that had diverse methodologies or philosophies so that researchers could begin to target successful approaches to preschool education. The Montessori method was often chosen as one of the programs for study since it had a long history and seemed different from typical preschool programs. For example, one researcher reported that of four programs studied, the Montessori philosophy was the only one that believed that children are "naturally curious, innately eager to learn, and capable of intense concentration" (Miller & Dyer, 1975, p. 5). Most of the research on the Montessori method was done in this era, and most of these researchers used subjects who attended one year of Montessori preschool and who were then sometimes followed through elementary school and even high school. The following sections describe short-term studies of the cognitive, affective, and social skills and abilities of Montessori preschool students, and relates findings of several large longitudinal studies.

Cognitive measures. Various methods have been used to assess the effectiveness of Montessori preschool programs. Some address subject area skills, such as reading or arithmetic, while others address more general tasks associated with achievement, such as Piagetian tasks or psychomotor skills. Bereiter (1967) found short-term gains for low socioeconomic students on the Wide Range Achievement Test that favored a direct instruction approach over Montessori. In a study of similarly disadvantaged students, Berger (1969) found that Montessori students had greater perceptual and cognitive skills versus other students in a traditional preschool. No difference in language skills was found in the Berger study. Similarly, Prendergast (1969) found no differences in prereading skills (eye-hand coordination, visual and auditory discrimination, and receptive language development) of Montessori and traditional nursery school students from middle-class homes.

Research on skills thought necessary for future success in mathematics and science has shown general trends in favor of Montessori methods. Savage (1973) concluded that students in Montessori classrooms for six months showed a pattern of higher success on Piagetian concepts of classification, number, measurement, and space and seriation versus students in a traditional setting. Two later studies (Morgan, 1978; White, Yussen, & Docherty, 1976) confirmed Savage's results of Montessori students' higher achievement on seriation and classification tasks, but not for other Piagetian tasks. Assessment of arithmetic skills in two studies found no significant differences for Montessori and traditional preschool students (Morgan, 1978; Stodolsky & Karlson, 1972), although Stodolsky and Karlson concluded that the Montessori preschool program enhanced sorting, visual-motor, and psychomotor skills. For science skills, Judge (1975) found no difference in the science observation skills of middle-class students in a Montessori preschool and those in Science-A-Process Approach, a program designed to teach these observation skills.

Measures of the cognitive skills of preschool students have generally shown results that favored Montessori, although these studies assessed short-term learning that may or may not have continued with these students. In a review of research on the Montessori method, Miezitis (1971) concluded that, because of the "intermediate degree of structure, encouraging independence within a structured environment, this method is more successful with disadvantaged children than are unstructured early-childhood-oriented programs" (p. 57). For the middle-class child, however, Miezitis believes that "the Montessori environment may be too restrictive in terms of experiences to challenge the child to experiment beyond concrete rules and conformity" (p. 57), a conclusion not shared by the researchers in the next section on affective and social effects of Montessori.

<u>Affective and social measures.</u> Because of the emphasis on the individual child and the greater use of manipulative materials in the Montessori philosophy, many researchers mistakenly believed that the Montessori method was concerned only with cognitive growth and overlooked the social and emotional growth of children (Berliner, 1974). As a result, many studies were undertaken to investigate these aspects.

In a pilot study Dreyer and Rigler (1969), who believed that the Montessori method was too concerned with cognitive growth, assessed the verbal and nonverbal creativity of middle-class Montessori and traditional preschool children using four instruments. The first task required children to tell about given objects. Montessori children gave more functional responses (told what the object was used for), while traditional students gave descriptions of physical characteristics. In a nonverbal task, scored by a test manual, children were asked to draw a picture on a blank piece of paper and to include a jelly-bean shaped figure in their drawing. Traditional students' mean score was 18.8 while Montessori students scored 10.8; it was concluded that traditional students were more creative on this task. On a third task children were asked to find an embedded figure within a picture. While there was no significant difference on the ability to find the figure, Montessori students performed the task faster. On the final task children were asked to draw pictures with no guidelines. On this task Montessori students drew significantly more geometric figures while the traditional students drew more people. The authors concluded from these tasks that

> Montessori children responded to the emphasis in their program upon the physical world and upon a definition of school as a place of work; the Nursery School children responded on their part to the social emphasis and the opportunity for spontaneous expression of feeling...The examiner reported that the Montessori children were highly task oriented, while for the Nursery School children it was an opportunity to be involved socially with the examiner. (p. 415)

They thus concluded that traditional students performed significantly better than Montessori students on these particular nonverbal measures of creativity.

In a dissertation designed to compare Montessori and Open-Education students in the creative abilities of fluency, originality, and imagination, Rush (1985) found no overall differences in the 115 middle-class 3- and 5-year-olds. Similarly, Brophy and Choquette (1973) found no overall differences on creative uses for specific items of students aged 4 and 5 attending two Montessori and two traditional preschools. This research was undertaken on the assumption that Montessori teachers may inhibit the creativeness of children by their demonstration of the "correct" use of Montessori materials, a hypothesis that was not confirmed.

The study of the social interaction skills of Montessori students has been shown to favor Montessori students in several studies. Berk (1973) considered Montessori a structured environment that stressed individual accomplishments and hypothesized that this environment may interfere with peer interaction. This hypothesis was not confirmed, and in fact, of the six types of preschools studied by Berk, the Montessori students had the highest incidence of peer interactions. Another study of interaction patterns also found that of three types of middle-class preschool situations, the Montessori classroom had the highest incidence of peer interactions and lowest incidence of adult-child interactions (Reuter & Yunik, 1973). Unfortunately, these authors made comparisons of classrooms having widely different adult-child ratios, which may have been the cause of the interaction differences. Murphy and Goldner (1976) replicated the Reuter and Yunik study with classrooms having the same adult-child ratio and failed to find significantly larger peer interactions versus adult-child interactions for the different preschools. This study did find, however, that Montessori children had longer interaction times within peer interactions, a finding the authors claim "may reflect a higher quality of social interaction in this setting. Longer interactions require more verbal ability and capacity for cooperation" (Murphy & Goldner, 1976, p. 727).

Finally, in a study of the sociomoral development of children, DeVries and Goncu (1988) observed the interaction patterns of pairs of children from Montessori and constructivist programs and found better social negotiation and conflict strategies for the constructivist pairs than for the Montessori pairs. They concluded that

> Sociomoral development in young children may be affected positively by preschool experience such as that provided by constructivist education with its emphasis on child autonomy and rich interpersonal experience. In contrast, experience in a

> quiet, orderly classroom controlled by teacher authority with little social interaction may negatively affect sociomoral development. (p. 25)

This description of a Montessori classroom as "controlled by teacher authority with little social interaction" does not agree with previous studies or literature on the Montessori method and leads one to believe that the authors had not studied a "typical" Montessori classroom, a problem discussed at the end of this review of Montessori research.

Longitudinal studies. The two studies that stand out as exemplary among research of preschool effects are longitudinal research programs by Karnes and her colleagues (Karnes & Hodgins, 1969; Karnes & Johnson, 1986; Karnes, Shwedel, & Williams, 1983) and Miller and her colleagues (Miller & Bizzell, 1983a, 1983b, 1984; Miller & Dyer, 1970, 1971, 1975). Both of these studies followed a large number of low-income students in several different one-year preschool programs (n=246 in four programs for the Miller studies and n=123 in five programs for the Karnes studies) through high school, focused on achievement and cognitive skills, and compared results to control groups.

The Karnes study initially reported smaller gains for the 4-year-old Montessori children than for children in the other four programs (Karnes & Hodgins, 1969). Follow-up studies showed no significant differences in achievement scores for students from the five programs during middle school and high school, although significant differences did exist favoring students who had spent one school year in preschool as compared to students having no preschool before their kindergarten year. Although Montessori students in this study never showed outstanding test scores during their school careers, they did have the highest success ratings and graduation rates at the end of high school of all of the experimental and control groups (Karnes, Shwedel, & Williams, 1983). The authors theorized that

> The Montessori emphasis on working independently and persistently to task may have transferred to classroom activities that these students encountered throughout their educational careers. These students were not necessarily intellectually

> advanced; in fact, among the five intervention groups they had the second lowest IQ scores at age 16... Nevertheless, the ability to work independently and to persist would certainly benefit students who need to spend longer periods of time to master course material. (pp. 161-162)

The other large-scale study, begun by Miller and Dyer (1971), also used low-income four-year-old children as participants. In this study, sometimes called the Louisville experiment, 246 students were randomly placed into one of four Head Start programs (Bereiter-Engelmann, Darcee, Montessori, or traditional) which were closely monitored for adherence to program doctrines by trainers from each program. Unfortunately only two Montessori classrooms were used in the study while four classrooms of each of the other types were used, because only two Montessori teachers could be located to teach the classes. These Montessori teachers were significantly different from the other Head Start teachers in age, experience, educational background, race, IQ, and personality variables. The two Montessori teachers were inexperienced and had fewer days of training than other teachers recruited for the other programs. Nonetheless, this study has been applauded (White, 1975) as exemplary.

Unlike the Karnes studies, achievement scores of the Montessori students in the Miller studies first leveled off but then rose sharply as compared with students from other preschool programs (Miller & Dyer, 1975). By grade 6, Montessori students had significantly higher reading and mathematics scores and IQ (Miller & Bizzell, 1983a). The authors commented that

> It is somewhat astonishing to find that sixth-grade children who were randomly assigned to a particular program at age 4, who were not brighter or more advantaged to begin with, and who had been scattered throughout a large school system during the first six years of school were performing significantly better at middle school than their peers who had experienced other types of prekindergarten programs. (p. 739)

When gender was taken into consideration, however, it was found that the reading and mathematics scores for Montessori boys were pulling up the means for the Montessori group and that Montessori girls were neither highest nor lowest among the groups (Miller & Bizzell, 1983b). The authors hypothesized that "when this sample entered prekindergarten, the little girls were more ready to process information gained through observation...while the little boys needed more kinesthetic methods of instruction" (p. 1586) and hence, the boys may have benefited more from the Montessori materials. This may be indicative of the importance for students, especially girls, to begin the Montessori method at the age of 3 in order to benefit most from the method. One final comment about this longitudinal study: the authors claimed that the individualized nature of the Montessori method may have made a difference in the results, since the method may help one student to excel at one subject while another student excels at something different. Since only group means were used in comparing groups, substantial changes for individual students would not have been detected.

Other longitudinal studies comparing preschool programs have found significant differences on measures of curiosity and divergent thinking favoring Montessori children after a six-year follow up study (Banta, 1969, 1970) and nonsignificant, but favorable, differences on all subtests of the Metropolitan Achievement Test favoring Montessori children after a nine-year follow up study (Sciarra & Dorsey, 1974, 1976). Unfortunately, these studies had a large attrition rate as the studies followed the children through middle and high school and may not reflect accurate conclusions about the entire beginning samples.

Montessori Elementary Studies

Only a limited number of studies addressing the Montessori method for elementary, middle, or high school compared to those for preschool were found for this review. One reason could be the vast number of preschool studies initiated at the beginning of the Head Start program, while there was not a similar interest at the elementary school level. Another reason may be the large number of Montessori preschools as compared to

Montessori elementary schools in the United States. Whatever the reason, only a handful of studies were found and reviewed here—an indication that more research on this method at the elementary level needs to be done. Most studies of the elementary Montessori method, as with the preschool studies, compared elementary Montessori students with those in "traditional" public schools. Unfortunately, the use of the term traditional was never fully defined. These comparisons were usually in the form of standardized achievement test results and most were recent studies.

The oldest study located compared patterns of task involvement, physical movement, and social interaction of 60 first- and fourth graders in Montessori and traditional schools (Baines & Snortum, 1973). These investigators found that 90% of traditional students' academic time was spent under the supervision of the teacher as compared to 1% for Montessori first graders and 14% for Montessori fourth graders. Other activities for Montessori students included individualized tasks, small group tasks, or teaching and being taught by other students. In both types of classrooms, students spent about 63% of their time focused on academic tasks. The authors summarize their findings in this way:

> The overall findings of this study cannot be taken as demonstrating the superiority of one educational system over the other. If the value judgment is made that children of a given age should acquire certain standard skills and content as the best preparation for participation in adult society, then it appears that the public school teachers in this investigation are effectively serving this need. On the other hand, if the goal is to develop self-direction (even at the risk of an 'uneven' rate of acquisition of standard skills), then the Montessori system is to be preferred. (p. 316)

This conclusion makes an important point regarding value judgments about what is considered a "successful" educational program for students, an idea that will be discussed in the conclusion.

Two dissertation studies of Montessori and traditional students found small differences or no differences on various measures of the effectiveness of programs

(Cisneros, 1994; Fero, 1997). Cisneros found no significant differences in 95 third-grade students from Montessori and traditional public schools on measures of attendance, grade promotion, mathematics, self-concept, or parental involvement. Fero found limited evidence that Montessori students scored higher than traditional students on achievement tests in various subjects. For second graders, traditional students scored higher on mathematics computation, concepts, and application, while fifth-grade Montessori students scored significantly higher in language expression and social studies. When Fero compared the aptitude tests of all 120 second- through fifth-grade students in the study on aptitude tests, Montessori students scored significantly higher.

In another study to evaluate the success of a Montessori public school, Moore (1991) compared standardized test results of enrolled students (age 3 through grade 3). Scores on the Iowa Tests of Basic Skills for first and second graders were above district and national norms, while third-grade students scored above district, and below national, norms on the Missouri Mastery and Achievement Tests. Questionnaires given to parents and students revealed that participants were satisfied with most aspects of the program.

One study dealt with the emotional impact of a Montessori program. Lee (1995) found that while the emotional well-being of six troubled high school students left in a traditional setting was unchanged or indicated negative trends, twelve students placed in a Montessori high school "felt more comfortable with social interactions within the school and more positive about the likelihood of academic success" (p. 3511). Both qualitative and quantitative methods were used in this dissertation.

Finally, one longitudinal study of current and former students from a Montessori school in Oregon was located. This study (Glenn, 1993, 1996) began in 1986, has followed participating students for ten years, and has included students who have stayed in the Montessori school or have left for other private or public schools. Originally 198 participants were included in the sample. Included in the third and fourth follow-up studies were 145 and 82 participants, respectively. The author believes that no significant differences exist between continuing participants and those who dropped out of the study.

Data for this longitudinal study included standardized test scores; student, parent, and teacher questionnaires; school grades; and personality measures. The third cycle of follow-up data (Glenn, 1993) found that current and former students of this particular Montessori school tested at or above national norms on standardized tests. Grades from 16 students who had left the school to attend a new school were found to have a mean of 3.38 out of 4.0. In the fourth cycle Glenn (1996) found that this mean rose to 3.48 and former students had mean percentile scores of 78th and 69th percentile for the verbal and mathematics portion of the SAT, respectively. At that time, the current students had mean percentile scores of 81st and 73rd percentile. Glenn (1993, 1996) found only limited support for his first hypothesis that the number of years in this Montessori school would be positively correlated with qualities emphasized in a Montessori environment such as relating well with peers. However, the second hypothesis, that Montessori participants would be at least as successful as the general population received considerable support (Glenn, 1993, 1996). Unfortunately, since these reports had a high attrition rate, these conclusions must remain tentative.

Summary of Montessori Research

Care must be taken in the interpretation of research on the Montessori method. As Schapiro (1993) points out, "the goals of a Montessori-based education go beyond the easily measurable" (p. 43). It is important to point out that in the process of research, assumptions are made about what constitutes a "good" or "successful" educational program. Missing from research in this area are qualitative or descriptive studies that could more deeply describe the effects of a Montessori education, such as possible deeper cognitive understanding of various elementary school topics, such as place value ideas. Also, when reading some of the research reports, descriptions of the classrooms used in the study are missing. Judgments about methods labeled "traditional" or "Montessori" are difficult if it is not clear to what those labels refer. Also, any school may call itself a Montessori school, because the name Montessori is not patented (Cohen, 1990). As in the DeVries and Goncu study (1988), a classroom labeled Montessori may not seem to embody the traditional Montessori ideals.

Keeping these cautions in mind, longitudinal and short-term studies of Montessori children show mixed results. In most cases, Montessori students scored similarly or higher on cognitive, affective, and social measures than comparison groups. In some cases, studies found significant differences on achievement tests or affective measures favoring Montessori students. However, the most conclusive results are long-term findings indicating that the Montessori method, which values independence and self-motivation, may cultivate these facilities and help disadvantaged preschool students overcome economically-deprived situations.

Significance of the Current Study

Most of the studies of the mathematics abilities of Montessori students reviewed here used Piagetian tasks for preschoolers and achievement test scores for students in elementary, middle, and high school. None of these studies provided an in-depth look at the understandings of Montessori students. The current study endeavored to determine how the prepared environment of a Montessori classroom, with its manipulative materials, its carefully sequenced tasks, and its focus on individual accomplishment, affected students' deeper understanding of mathematics, and more specifically, place value concepts.

Research on place value has been reviewed to provide a picture of what is known about students' learning of place value so far. Most of this research has been conducted with students in traditional classrooms, focusing on large group instruction, or in social-constructivist classrooms, focusing on small group problem-solving instruction. The current study concentrated on students who experienced instruction focusing on individualized, student-directed classrooms containing numerous manipulative materials, in order to compare these results with those from another type of learning situation.

CHAPTER 3
METHODOLOGY

The current study was an exploration to investigate the principles put into place in the classroom by Montessori as compared to those in place in a comparison school, specifically in the area of place value concepts. This chapter describes the setting and subjects of the study and data collection techniques. Since no studies were found that investigated Montessori students' understanding of place value concepts, descriptive methods were used to illuminate these understandings, and in order to generate hypotheses for further study. Clinical interviews were used to gather information about students' thinking about place value concepts.

The following specific questions guided the study:

What are the differences or similarities in the way that Montessori students complete tasks involving place value concepts as compared to non-Montessori students in first through third grade?

(a) *What are the differences or similarities in the way that Montessori students complete tasks involving counting as compared to non-Montessori students in first through third grade?*

(b) *What are the differences or similarities in the way that Montessori students complete tasks involving mental addition and subtraction as compared to non-Montessori students in first through third grade?*

(c) *What are the differences or similarities in the way that Montessori students complete tasks involving the forming and unpacking of hundreds and* *thousands as compared to non-Montessori students in first through third grade?*

(d) *What are the differences or similarities in the way that Montessori students complete symbolic tasks as compared to non-Montessori students in first through third grade?*

Setting and Population

Two Catholic schools, a Montessori and a comparison (non-Montessori) school, were chosen from schools in the Columbus, Ohio metropolitan area. Classroom observations and interviews were conducted to describe, in chapter 4, the specific classroom environment and teaching methods in the schools. Many students from the Montessori school live in the suburbs of Columbus and the school offers scholarships to approximately 10% of its students. This school is affiliated with the Columbus Diocese, but is operated independently with its own Board of Trustees. The tuition rate is higher that of other Catholic elementary schools in the area. The author sought a private, non-Montessori school that would have students of comparable socioeconomic level to these Montessori students. A parish school in a Columbus suburb was chosen for comparison because the students in that school, like the Montessori students, reside mainly in the suburbs of Columbus rather than in the city itself.

Students in this study were first, second, and third graders. These age levels were chosen because the concepts that develop from place value and number concept instruction are just beginning to emerge after one year of instruction (Jones, Thornton, and Putt, 1994). Therefore, the author could be certain that in both schools first-grade students have had some experiences thinking about place value, and the age and experience range of first through third grade provides a good cross-sectional view of students' place value concepts in these particular settings.

Research Methods

Clinical Interviews

A clinical interview, a method developed by Piaget and modeled after psychiatrists' methods of diagnosis, is a one-on-one situation in which a subject is presented with a task so that the interviewer can glean information about the subject's thinking (Ginsburg, 1997). The main objective of a clinical interview is to "question the child at precisely those points where direct observation or preconceived tests would fail to get below the surface of cognitive content or functioning to reveal the underlying structural aspects" of his or her cognitive processes (Codd, 1981, p. 146). A clinical interview is not a teaching technique, but rather a method to find out, at a certain point in time, what a child knows.

In this study, clinical interviews were conducted to help the author gain an in-depth understanding of the mathematical knowledge of place value concepts of these students. Interview protocols remained basically the same for all students for several reasons. First, students at all levels were given the same opportunity to answer the questions with no premature decisions being made about the abilities of younger versus older students. Secondly, analysis of ability levels would be hindered if interviews were different for the different levels. However, interviews did vary slightly in that, if many questions were too easy for a student, a few more advanced questions of the same types were asked. Therefore, all interviews began with the same tasks and developed according to each student's responses and the author's in-the-moment analysis.

Interview Protocol

As discussed in the literature review, place value concepts consist of many components, including: counting, partitioning, grouping, and number relationships. The variety of tasks in the interview in this study attempted to address these components. Many of the tasks involved the use of three types of manipulative materials: Unifix cubes, strips and squares, and Cuisenaire rods. These materials were chosen because some of them would be familiar to some students, while others would be new materials. Specifically, the strips and squares materials used in some of the tasks were similar to the tens and ones pieces of the base-ten materials (golden beads) regularly found in Montessori classrooms. However, these materials were different enough from the traditional Montessori materials to pose a challenge to students. Some of the non-Montessori school students had worked with Unifix cubes, but the other materials were unfamiliar to most of them.

The following sections describe each task and the purpose for inclusion in the study. Appendix C contains all of the interview task questions. Figure 3.1 summarizes information about the source of the tasks and the research question(s) to which each will be applied.

The tasks themselves varied according to possible familiarity by the students. Both groups of students had or will encounter the counting exercise and the vertical addition task, depending upon grade level. Students in both schools may have encountered addition problems written horizontally, but would probably not be asked to solve them mentally. Many of the second-grade non-Montessori students mentioned building towers of ten using Unifix cubes in class, but were not asked to guess how many cubes were in a box or how to count them as in these interviews. Also, non-Montessori students may have been more familiar with symbolic tasks such as the number cards and digit cards tasks. The strips and squares counting and uncovering tasks may have been more familiar to Montessori students who worked with base-ten blocks on a regular basis. The other tasks, strips and squares hidden task, flower task, and bean task were assumed to be new to both groups.

Task Group	Research Question(s)	Task	Source
Counting	A A A A	Counting Unifix Cubes Strips and squares counting Strips and squares uncovering	Author Kamii, 1986 Cobb and Wheatley, 1988 Cobb and Wheatley, 1988
Mental addition and subtraction	B B	Strips and squares hidden task Horizontal addition	Cobb and Wheatley, 1988 Cobb and Wheatley, 1988
Forming and unpacking 100s and 1,000s	C C	Flower task Bean task	Jones et al, 1996 Author
Symbolic	D D D	Number cards Digit cards Vertical addition	Author Author Author

Figure 3.1: Task group and source and which research question answered

General questions. After explaining the terms of involvement in the study (see Appendix B), the student was asked four general questions to glean information about his background and classroom context (see Figure 3.2). These questions were also posed in order to put the child at ease before any questions involving mathematical content were asked.

General Questions
How long have you been here at this school?
Where did you go to kindergarten?
What do you do in math class?
What materials do you use in math class?

Figure 3.2: General questions

Counting task. Several questions involving the ability to count, the first component of place value ideas according to Jones and Thornton (Jones, Thornton, & Putt, 1994; Jones, Thornton, Putt, Hill, Mogill, Rich, & Van Zoest, 1996), were the first content questions in the interviews (see Figure 3.3). Since place value concepts grow out of counting schemes, the author wanted to establish the level of each child's ability to count. Also of interest were any differences in the rhythm of their counting, for example when counting by tens would a child stop at 100 or continue to 110 and beyond. These questions were also meant to be fairly easy for the child and to put her more at ease with the interview.

Counting task questions
How high can you count?
Would you start at 35 and count up?
Would you count by 10s?
Would you start at 60 and count by 10s?
Would you count by 100s?
Would you start at 700 and count forward by 100s?
Would you start at 24 and count backwards?
Would you start at 60 and count backwards by 10s?

Figure 3.3: Counting task questions

Unifix cubes task. These questions were adapted from Kamii (1986), who used plastic chips, in order to determine more information about children's counting schemes and abilities. In the current study approximately 70 Unifix cubes were used to allow the opportunity for children to snap the cubes together to form chains if they so desired. As in the Kamii study, children were first asked to guess the total number of cubes to ascertain how close they could guess the correct amount and if their answers would differ by grade level or school type (Figure 3.4). Next, each child was asked to count the cubes to determine how they were counted, for example by ones, by twos, or by tens. Finally, children were asked how they could place the cubes so that the next child could count them even faster. This question was asked to see if a child would think to place the cubes in rows of ten. Later in the interview, other questions were asked to determine if children were actually able to count by tens, whether or not they thought of this way to count in this particular part of the interview.

Unifix cubes task questions (approximately 70 cubes on the table) Would you guess how many are here? Would you count all these cubes for me? How could you put them so the next child could count them even faster?

Figure 3.4: Unifix cube task questions

Strips and squares task. Questions in this section were used because of their successful use in another study to uncover children's levels of understanding of ten as a unit, a vital concept in place value understanding (Cobb & Wheatley, 1988). Students who need to count individual squares on the strips are at a different developmental level than those who recognize a strip of ten squares as a 10.

The strips and squares tasks used strips of poster board upon which ten individual squares had been pasted (Figure 3.5). These materials were similar but not identical to materials regularly found in Montessori classrooms, specifically the base-ten materials. The first strips and squares task was posed to further investigate each child's counting methods when faced with materials in the form of tens and ones (Figure 3.6). Developmental levels can be determined by taking note of whether students count individual squares on the strips or can recognize a strip of ten squares as a 10.

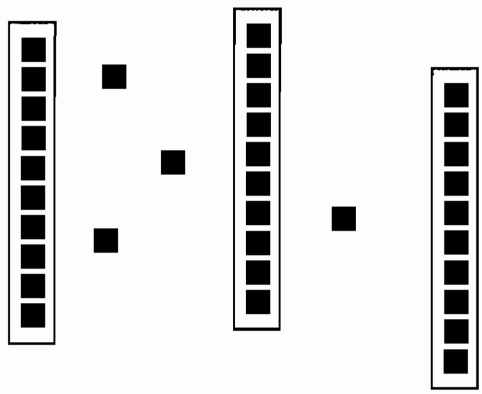


Figure 3.5: Strips and squares task materials

<u>Strips and Squares Task</u>: <u>*Counting*</u>
Would you put 13 squares on this paper?
Would you put 86 squares on this paper?
If I take off this (one strip) how many are left?
(If child did not use a "ten" for the 13) Would you put 18 squares on this paper?

Figure 3.6: Strips and squares counting task questions

The second task involved a board upon which many strips and squares were pasted and involved the use of addition and missing addend questions. The author began by hiding the board from view with a cloth and telling the child that he would be expected to say how many total squares he saw as the author uncovered the board a little at a time (see Figure 3.7): one strip (10); three squares (13); two strips (33); four squares (37); three squares (40); one strip (50); two squares (52); two strips (72).

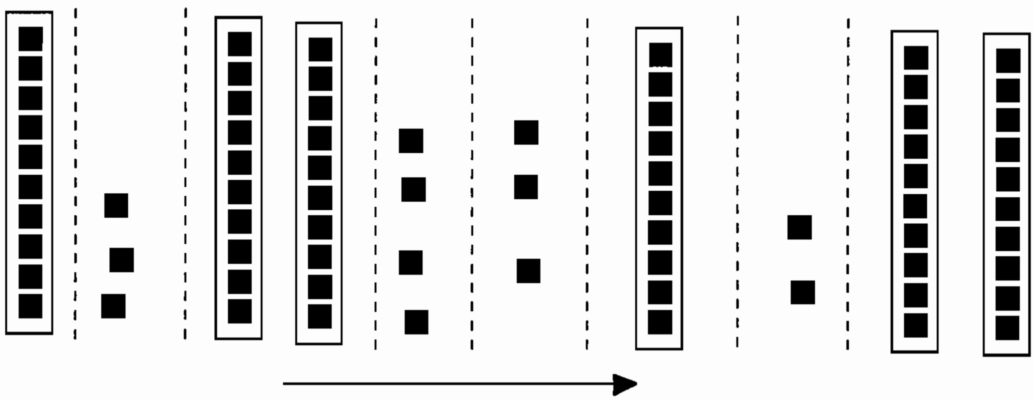


Figure 3.7: Uncovering task materials

This uncovering task was used to determine whether a child used concepts of ten to count the squares or if she counted all of them by ones. For example, a child might have *counted on* each time more squares were revealed (10, 13, 33, 37, ...), may have counted by ones entirely, or may have mixed up these strategies and counted all the ten bars by tens first and then counted on by ones.

The final task using the strips and squares involved hiding some of the materials under a cloth (Figure 3.8) and posing addition and missing addend tasks (Figure 3.9). Questions in this section became more difficult with addition tasks presented first and then missing addend questions later. As with the earlier strips and squares tasks, these questions were asked to categorize children's abilities to count and whether they used ten as a unit or merely counted by ones. The author hoped that presenting addition and subtraction tasks in this way, with materials, would promote thinking strategies and number sense as opposed to the use of the standard addition or subtraction algorithms. Therefore, in the case of a child who used the standard addition algorithm for later tasks involving horizontal or vertical addition problems, the author might find evidence that the child could indeed use thinking strategies in strips and squares task.

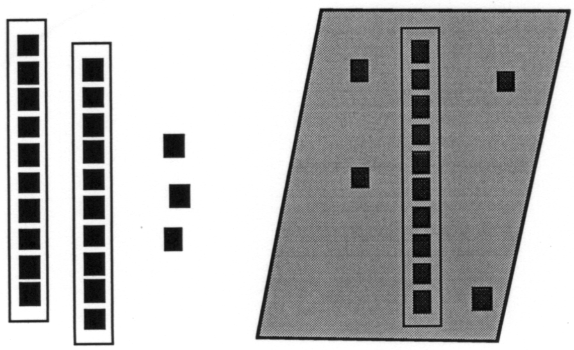


Figure 3.8: Strips and squares hidden task materials

<u>Strips and Squares Task: *Hidden tasks*</u>

Addition

(2 strips and 3 squares visible) "I've hidden 4 squares and 1 strip; how many are there all together?"

(3 strips and 8 squares visible) "I've hidden 4 squares and 1 strip; how many are there all together?"

Missing addend

(3 strips visible) "I've hidden some under here; If there are 50 altogether, how many are covered?

(2 strips and 4 squares visible) "I've hidden some under here; If there are 36 in all, how many are covered?"

(2 strips and 3 squares visible) "I've hidden some under here; If there are 41 in all, how many are covered?"

Figure 3.9: Strips and squares hidden task questions

Cuisenaire rods task. Cuisenaire rod materials have yellow cubes and many colored rods. The color of a rod depends upon its size: red rods are the equivalent of two cubes put together, the green rods are the equivalent of three cubes (Figure 3.10). Cuisenaire rods are smooth—there are no ridges to indicate the number that the rod represents. The questions in this section of the interview were asked to determine whether a child could combine rods to form the "length" of 12 in various ways and could replace a "three" rod with a "two" rod in order to get a total of 11.

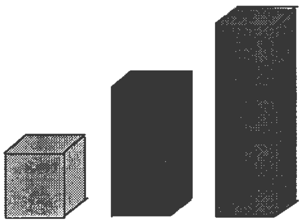


Figure 3.10: Cuisenaire rods task material

Cuisenaire rods Ask child to verify that the red rods are made from 2 unit cubes and the green are made from 3. "Please put the equivalent of 12 cubes in this bowl." "Can you show another way? Another way?" "What about 11 squares?"

Figure 3.11: Cuisenaire rods task questions

Horizontal addition task. Cobb and Wheatley (1988) found that students solved the two problems in Figure 3.12 in different ways. Even though most adults would see these problems as the same, many children in their study used their own invented algorithms for problem *a* and got the answer correct, but used school-taught procedures incorrectly to solve problem *b*. Moreover, the students were not disturbed by getting two different answers. Therefore, to allow students the best possibility of using their own invented strategies for solving addition problems with regrouping, the students in this study were asked to mentally solve four horizontal addition problems (Figure 3.13) so that the author could look for students' own solution strategies. Vertical addition problems were posed later in the interview for comparison purposes. These tasks were included because of the heavy emphasis on paper-and-pencil procedures in traditional classrooms as a sole indicator of place value understanding. It was predicted that the non-Montessori students would do well on problems involving computation, but it was unclear whether they would or could use their own thinking strategies.

(a) 28 + 13 = ____ Horizontal placement	(b) 28 +13 Vertical placement

Figure 3.12: Horizontal and vertical placement of addition tasks

16 + 9 =
28 + 13 =
38 + 24 =
39 + 53 =

Figure 3.13: Horizontal addition tasks

Flower task. This problem was adapted from Jones et al. (1996) to investigate a component of place value called grouping. The purpose of the question was to determine whether students could put tens together into hundreds or multiples of ten in order to extend the idea of place value into hundreds. For instance, many children can count by tens to 100 but may not be able to immediately say that 10 tens is 100.

Flower task questions
"What if you went outside to pick some flowers for (principal's name). Each flower has 10 petals. You found 2 loose petals on the ground and picked 10 whole flowers. How many petals would you have?"
If child is unable to answer, she is asked about 3 loose petals and 4 whole flowers.

Figure 3.14: Flower task questions

Bean task. This problem was intended to complement the flower problem and involved partitioning, another component of place value described by Jones and Thornton (Jones, Thornton, & Putt, 1994; Jones, Thornton, Putt, Hill, Mogill, Rich, & Van Zoest, 1996). Instead of determining if students could create tens and hundreds from ones, this problem (Figure 3.15) asks students to *unpack* tens and hundreds and even thousands. For example, if a child was asked how many tens were in the number 123, the child would probably point to the 2 in the ten's place. However, the idea that there are a full 12 tens in the number 123 may escape a child's notice unless he understands place value concepts for hundreds and thousands. Because students may be familiar with the terminology of "ten's place" or "hundred's place" without knowing the meaning of the concepts, the bean questions were worded to uncover students' understanding of place value for larger numbers.

Beans in a row (16 beans are placed upon the table: 10 in a row and the six leftovers in a pile) "If you had beans like these, and you put beans in rows of 10, how many full rows of 10 beans could you make from 35 beans?" "from 72 beans?" "from 123 beans?" "If you put 100 beans in a cup, how many cups could you fill from 423 beans?" "from 1,214 beans?"

Figure 3.15: Bean task questions

Symbolic tasks. Whereas other tasks in the interview involved orally stating numbers, the tasks in this final section of the interview (Figure 3.16) involved written numerals and so were labeled symbolic tasks.

The first two of the three tasks are similar. In the number cards task, the child is shown two cards upon which are written two numbers with three or four digits (four digits for older children or for younger children who showed higher ability in the first parts of the interview). The child is asked to choose the larger number and to explain why it is larger. The numbers were chosen so that, if a child compared each digit without looking at the number as a whole, the smaller number may seem larger. For instance, though 372 is larger than 358, one of the digits in 358 is larger than those in 372, so a child may choose 358 as the larger.

In the digit cards task the child is requested to create the largest three- or four-digit number possible from several cards with digits on them. This task was included to supplement the information from the number cards task.

The last task was included to determine the solution strategy for vertical addition problems to be compared with solutions to the horizontal problems. Large numbers were included to determine the limit of the child's ability to add increasingly large numbers. At first, the child was asked to guess what was the most difficult problem the child could do and the author began with that problem when asking the child to actually add the numbers. This was done to save time at the very end of the interview, so that the author did not have to ask each child to start at the beginning of the list.

Symbolic tasks

Number Cards

Show card with 1,247 and 1,326. "Which is bigger?" "Why?"
Show card with 358 and 372. "Which is bigger?" "Why?"

Digit cards

Show cards with digits written on them: 1, 3, 4, 5. Show that 1 next to the 3 makes a 13.
"What is the biggest number you can make with these cards? Can you make an even bigger number?"

Vertical Addition

Student is asked which problems they can do and then requested to pick hardest on their list and solve. Students are permitted to use a pencil.

13	125	2,474	68,183	134,023
+ 48	+ 492	+ 5,823	+ 27,207	+ 245,912

Figure 3.16: Symbolic tasks questions

Supporting Information

Informal classroom observations were made as the author interacted with the children and teacher throughout the study. Interviews with participant teachers and principals were conducted and documents were collected at the end of the study to determine the methods of mathematics instruction, mathematical tasks regularly given to students, time devoted to mathematics, social atmosphere of the classroom, and materials used in the teaching of mathematics in classrooms of both schools. Observations and interviews were used to provide impressionistic descriptions of the classrooms for the reader.

Procedures

Sample

Permission forms were sent home during autumn with all first-, second-, and third-grade students in both schools. Those students who returned their signed permission slips formed the pool of students (Table 3.1). During autumn quarter, several students from each school were interviewed for pilot data information. These students were not interviewed again for the dissertation study.

At the non-Montessori school, about twice as many students returned forms than were needed for the study, so approximately eight students (four boys and four girls) were randomly chosen from the pool of students in each of the two classrooms per grade level, for a total of about 16 students per grade level (see Table 3.1). Two of the third-grade students were later dropped from the study due to inaudible videotapes. At the Montessori school, which has fewer students in each grade level, 57 permission forms were returned, but one student refused the interview, one student moved away, two students were interviewed but were found to have just begun Montessori education that year and were dropped from the study, and six students were interviewed for the pilot study.

Classroom	Grade	Number of teachers in classroom	Number of students in classroom	Number of permission forms returned	Number of students interviewed
Montessori classroom A	1	3	45 Total	10	6
Montessori classroom A	2			11	9
Montessori classroom A	3			10	8
Montessori classroom B	1	2	30 Total	9	8
Montessori classroom B	2			8	8
Montessori classroom B	3			9	8
non-Mont classroom A	1	1	30	18	7
non-Mont classroom B	1	1	30	17	9
non-Mont classroom C	2	1	30	18	8
non-Mont classroom D	2	1	30	12	8
non-Mont classroom E	3	1	30	14	6
non-Mont classroom F	3	1	30	15	8

Figure 3.17: Number of students participating in study by classroom

<u>Student Interviews</u>

After exclusions, participants were individually interviewed during January, February, or March. Each interview lasted approximately 30 minutes and was videotaped and audiotaped. Interviews at the non-Montessori school were held in a fairly large storage room equipped with a large table and chairs. This room was located very near the children's classrooms, and the interviews were never interrupted. At the Montessori school the only available space was the faculty lounge containing a large table and chair. This room was fairly well used by faculty during the day, so interviews were interrupted once or twice each. Students did not seem particularly bothered by these interruptions and continued to talk.

Interviews were conducted approximately three days per week. Two of those days were spent at one school and one day at the other school. The following week that pattern was switched. Teachers were notified by phone or notes which students would be interviewed from their classrooms that week. Depending upon the schedule of the school and if any special events were occurring, approximately six interviews were carried out on any particular day. The author made a concerted effort not to interview students of one kind (e.g., same grade in the same school) all at once. Realizing that over a three-month period the interviewing style might have changed slightly and the curriculum in the schools was forging ahead with children learning and developing more knowledge and skills, the author wanted to evenly spread out the interviews.

During the interviewing period, the author determined that the Cuisenaire rods questions, focusing on counting, were not eliciting interesting or substantial results. Therefore, these questions were dropped from the protocol approximately halfway through. At the same time, the researcher noted that additional clarification for the vertical addition questions was needed. Therefore, the horizontal addition questions were added. These changes resulted in revealing significant differences between Montessori and non-Montessori participants on the horizontal addition task.

<u>Teacher and Principal Interviews</u>

Interviews with teachers and principals were held informally at the end of the school year and in the summer. The main purpose of the interviews was to gather information to describe the context of the student learning in order to briefly describe the setting to the reader. Another purpose of the interviews was to show the teachers the interview materials. Teachers and principals were also given a general idea of their students' performance without using names. Interviews lasted approximately 30-45 minutes, were held in classrooms or offices, and were audiotaped when possible. At the non-Montessori school, each teacher in the first through third-grade classrooms and the principal participated. At the Montessori school, the teachers responsible for the mathematics portion of the curriculum (one teacher from each of the two classrooms) and the principal participated.

CHAPTER 4
DATA ANALYSIS

Having already discussed the subjects of the study and methods of research in chapter 3, this chapter provides a summary of the data collected. First is a description of the context of the children's learning—the classroom culture, tasks, tools, and discussion—gleaned from interviews with the classroom teachers, principals, and students. Following that description is the analysis and summary of the task responses of the first-, second-, and third-grade Montessori and non-Montessori students.

Comparison of the Classrooms

Data gathered through classroom observations and interviews with classroom teachers and principals of the Montessori and non-Montessori schools were described using the framework described in chapter 1 (Hiebert et al., 1997). This framework breaks the classroom down into five aspects: tasks, role of the teacher, tools, classroom culture, and equity. This breakdown is provided to inform the reader of the context of the children's learning. First is a chart of mathematics topics covered in the first three grades of the Montessori and non-Montessori schools.

Figures 4.1 through 4.3 describe the diocesan course of study for Catholic schools in the region and the curriculum followed by the Montessori and non-Montessori schools. The non-Montessori school closely follows the textbooks and workbooks for their curriculum (grade 1: Math Advantage by Harcourt-Brace, grades 2 and 3: Math in My World by MacGraw-Hill), and so the chart for the non-Montessori school is a summary of the textbook and workbook chapter headings. The teachers at the Montessori school wrote their own curriculum for mathematics; therefore, the chart for the Montessori curriculum is a summary of their listed course of study topics.

As can be seen from the figures, the non-Montessori school curriculum closely follows the diocesan course of study for all three grade levels in place value and related topics. In first grade, the curriculum includes sequencing numbers to 100, skip counting, ordering numbers, strategies for addition of single digit numbers, and mastering basic addition and subtraction facts. Addition of multidigit numbers without regrouping using manipulative materials and symbols is mentioned in the diocese course of study for grade 1, but the textbook chapters do not contain this topic. Second grade includes extensions of these activities to addition of two- and three-digit numbers with and without regrouping and the beginnings of multiplication. The curriculum of grade 3 includes numbers to 100,000 and extensions of multiplication and division up to division with remainders.

The Montessori curriculum differs significantly from the curriculum in the non-Montessori school. In first grade multidigit addition with and without regrouping is covered as well as numbers through 1,000, squaring numbers, and nonregrouping multiplication and division up to four digits, topics included in the non-Montessori second- and third-grade curriculum. The curriculum of grade 2 includes extensions of the four arithmetic operations and word problems, found in the third grade in the non-Montessori school. The third grade curriculum of the Montessori school calls for numbers up to one billion, multistep word problems, and extensions of multiplication and division to include division with three-digit divisors.

The diocesan course of study mentions using manipulative materials various times in their topics list. The author does not have information about the teachers' guide to the textbooks and workbooks used in the non-Montessori school, and therefore cannot make comments about the expected use of manipulatives for these textbooks. The Montessori course of study lists the topics of study in one column and then in another lists the manipulative material to be used for the teaching of this topic. This is done for every topic.

Diocese Course of Study Grade 1	Non-Montessori Curriculum Grade 1	Montessori Curriculum Age 6
	Counting	
read and write numbers in and out of sequence through 100 count backwards from 10 skip count by 2s, 5s, and 10s to 100 order and compare numbers concepts of >, <, and = ordinal numbers (1st, ... 10th)	one-to-one correspondence skip count by 2s, 5s, and 10s to 100 order and compare numbers through 100 concepts of >, <, and = ordinal numbers (1st, ... 10th) building numbers to 100 even and odd numbers	identify numbers to 1,000 concepts of >, <, and = skip count 1-10 to the square of the number write numerals to 5 digits (10,000s) learn the first law of the decimal system (regrouping/exchanging by 10s) study expanded notation to 1,000 even and odd numbers
	Addition and Subtraction	
combine and separate numbers strategies of adding of whole numbers such as counting all, counting on, one more and one less, doubles, doubles plus one, make ten, and using ten frames begin to identify fact families describe operations of +/- in words compute sums and differences horizontally and vertically model, explain, and master basic +/- facts (0-10) make a reasonable estimate for +/- statements demonstrate +/- of two-digit and three-digit numbers with no regrouping using manipulatives and symbols	+/- story problems writing +/- sentences +/- facts (0-10) strategies of addition: counting on 1, 2, and 3; doubles three addends strategies of subtraction: counting back 1, 2, and 3; 0 relating addition and subtraction	+/- in vertical form up to 4 digits static and dynamic (nonregrouping and regrouping) memorize addition facts 0-5, added to 0-10 develop and explain strategies to solve subtraction word problems missing addend problems commutative and associative properties for addition
	Multiplication and Division	
(not addressed in Diocese Course of Study)	(not addressed in non-Montessori curriculum)	identify and explain multiplication and division multiply and divide in vertical form up to 4 digits static (nonregrouping); 1-digit divisor (dynamic) begin memorization of multiplication facts 0-5, multiplied with 0-10 learn concept of squaring of numbers 1-10

Figure 4.1: Comparison of Montessori and non-Montessori place value and related curriculum for grade 1

Diocese Course of Study Grade 2	Non-Montessori Curriculum Grade 2	Montessori Curriculum Age 7
	Counting	
correctly sequence whole numbers on the number line through 20 develop the concept of place value and find equivalent forms of numbers using concrete models of 100s, 10s, 1s write numbers through 200 in and out of sequence read numbers through 1,000 use ordinal numbers first through thirty-first even and odd numbers using concrete materials front-end estimation	count on; count back order numbers to 100 numbers to 1,000 skip count by 2s, 5s, 10s place value: 100s, 10s, 1s	form quantities and identify numbers to 1,000,000 identify odd and even numbers expanded notation to 1,000,000s
	Addition and Subtraction	
translate real-life situations involving +/- into conventional symbol basic +/- facts to 18 recognize +/- as inverses compute +/- horizontally and vertically add two- and three-digit numbers with one regrouping subtract a two-digit number from a three-digit number with no regrouping subtraction two-digit numbers with regrouping mental +/-	three addends add and subtract 9; make a 10; subtract with doubles; fact families two- and three-digit +/- with and without regrouping	refine addition skills with static and dynamic addition units to 1,000s memorize addition facts 1-10 +/- word problems static subtraction; dynamic subtraction exchanging in 1 place inverse relationship of +/- find missing elements in subtraction problems
	Multiplication and Division	
explore multiplication and division concepts by joining and separating equivalent sets of objects	multiply by 1, 2, 3, 4, and 5 how many groups; how many in each group	multiplication with 1-digit multiplier (static and dynamic) continue to memorize multiplication and division facts 1-10 multiplication and division word problems dynamic division in horizontal form with 1-digit divisor find missing elements in multiplication problems

Figure 4.2: Comparison of Montessori and non-Montessori place value and related curriculum for grade 2

Diocese Course of Study Grade 3	Non-Montessori Curriculum Grade 3	Montessori Curriculum Age 8
Counting		
read, write, and order whole numbers to 1,000 round to 10s and 100s place place value through 100,000s using manipulatives	numbers to 100,000s	identify numbers and expanded notation to billion (1,000,000,000) write numerals to 12 digits
Addition and Subtraction		
mental +/- by finding groups of 10 estimate addition +/- two-, three-, and four-digit numbers with one or more regrouping	+/- facts +/- two- and three-digit numbers with regrouping	add columns of one-, two-, and three-digit numbers add addends of various lengths with random exchanging multistep +/- word problems master +/- facts subtract in vertical and horizontal form for mixed digits
Multiplication and Division		
model multiplication and division using manipulatives describe multiplication and division in words and use conventional symbols relate multiplication to skip counting recall multiplication and division facts through 12x12 using strategies translate real-life situations into symbols relate even numbers to division by 2 multiply two- or three-digit number by a one-digit number with regrouping division with one divisor with or without remainders round factors and use multiples of ten to estimate products	equal groups multiplication facts for 0-9; multiplication table 0-9 three or more factors squares division by 1-9 division with remainders	multiply in vertical and horizontal forms with 1-digit multiplier and various digit multiplicand gain skill in process of multiplication with more than 1-digit multiplier memorize multiplication facts 0-5 and 10 division with two- and three-digit divisor continue memorization of division facts division word problems inverse relationship of multiplication and division distributive property

Figure 4.3: Comparison of Montessori and non-Montessori place value and related curriculum for grade 3

Montessori Classroom

The Montessori school is a private school with a 30-year history, which provides Montessori education to students from age 3 through grade 8. Most students in this school have had Montessori preschool experiences since age 3. In this school, keeping with the Montessori philosophy, preschool students aged 3-6 are grouped together as are students in first, second, and third grades. The school contains two classrooms for the first, second, and third graders; one classroom has 45 students with 3 teachers, while the other has 30 students with 2 teachers. The mathematics portion of the curriculum is overseen by one teacher in each classroom, each of whom has other duties as well. Interviews with these two teachers and the principal were conducted to provide more insight into the specific curriculum of this school.

A mathematics lesson in these classrooms is typically held three to four times per week for 30-45 minutes with groups of 4-10 students at a time. The teacher may begin by demonstrating a material or briefly explaining a game that would be based on a concept to be learned. Children are guided through the material or game with questions or prompts from the teacher or by asking questions of the teacher. Individual work is assigned, often involving the materials just demonstrated by the teacher. Students are assessed by showing the teacher their mathematics notebooks and by individually administered tests.

1. Tasks. Tasks given to students in the Montessori classrooms provide practice of the concept learned in the small-group lessons. Most of these tasks are written on teacher-prepared cards and are coded for the teacher as to the type of work involved, such as two-digit addition with regrouping or three-digit subtraction without regrouping. Children are allowed to choose when this practice occurs in their day, with some restrictions. One student preferred this method of work by comparing the activity at his previous school to his new Montessori school, "The last one you just sit and do work. I prefer to choose the type of ... my work" (Interview 2/5/99).

2. Role of the teacher. The role of the Montessori teacher is to keep track of what mathematics the children know and to guide the children to the next step. Montessori teachers are provided with large manuals containing one or more lessons for each concept with instructions for lesson delivery. When asked if these books require teachers to be more directive with children or otherwise limit the teacher, one teacher mentioned that even though "it's step by step in the albums, you have the freedom to let the child guide you more, because if in small groups, if two kids are getting it right on, you can bump them ahead" (Interview 9/9/99). The teachers also mentioned their role in connecting the ideas embodied in the different representations of the manipulative materials and a goal of helping students progress to the point of no longer needing materials.

3. Classroom culture. Walking into either of these Montessori classrooms, a visitor is likely to find children working everywhere, at tables, on the floor, on couches. A teacher in the Montessori classroom orchestrated the daily activities, and was a major resource for the children, but when the teacher was working with another student or group of students, children often asked for help from their peers or an older child. Doing most of the presentations in small groups allows for more discussion and questions from the students and, as one teacher said, "gives the child more independence and ownership" (Interview, 9/9/99).

4. Tools. In the Montessori classrooms, the major tools are manipulative materials, many of which were described in chapter 1. One teacher described the use of the Montessori materials: "The way we use materials, I feel they are math tools, not just objects to play with" (Interview 9/9/99). Materials are used to demonstrate a particular concept. For instance, one teacher described the use of the calculator to help the student write the numerical symbols for numbers such as 101. The student who already knows how to count to 1,000 may have trouble writing the digits of a number so large, sometimes

including too many or too few zeroes. Other tools include the use of discussion in small groups or individual lessons with the teacher.

5. Equity. When asked what happens in the case of a child who has difficulty, one teacher responded that they have many ways to experience the same concept, for instance, addition can be learned using dice, flash cards, and number lines. Another teacher mentioned that "if I have a kid who says 'I'll just sit here and watch,' it's like, no you don't. I'll do the lesson again and I need to have a response from you because I don't know if you know what's going on" (Interview 5/13/99).

Comparison Classroom

The non-Montessori school is a private, Catholic school which provides education to students from grade 1 through 8. The school is expanding by adding another classroom for each grade level year by year. At the time of this study, each grade from 1 through 5 had two classrooms of 30 students each, while the rest of the school had one classroom for each grade level. In the study classrooms, one teacher is assigned to 30 students and these teachers are responsible for teaching all academic subjects.

Interviews with the six teachers at the non-Montessori school assigned to the three grade levels, two each at the first-, second-, and third-grade level, were conducted at the end of the study. Each "team" of teachers was interviewed separately, that is, the first-grade teachers were interviewed together but separately from the second- and third-grade teachers.

A mathematics lesson in these classrooms is typically provided daily for approximately 30 to 40 minutes. Usually the lesson begins with the teacher demonstrating a skill and students are then asked to individually and quietly practice this skill in their texts and workbooks during this time. Sometimes students work together in pairs and sometimes manipulative materials are used. Students are assessed using tests, occasionally administered individually.

1. Tasks. In the non-Montessori classrooms, tasks given to students were problems similar to those explained by the teacher during the mathematics lesson. These problems, assigned for classwork and for homework, were done in textbooks similar to workbooks that had short answer or fill-in-the-blank types of tasks for students. These textbooks were chosen because they were closely aligned with the diocesan course of study for mathematics. Sometimes worksheets were given to practice skills learned in previous lessons.

2. Role of the teacher. The role of the teacher was to introduce a concept (sometimes with manipulatives), show a method, "explain a couple of things," and then assign textbook pages for this concept. A teacher mentioned that they "pretty much whole group teach" (Interview 5/12/99). Teachers also tested students, sometimes individually, on the material presented in class.

3. Classroom culture. In this school, teachers are the mathematics authorities. Methods and strategies of doing problems come mainly from the teacher, who lectures or sometimes demonstrates manipulative materials. Students in these classrooms are politely quiet and attentive. Students may work in pairs some of the time to complete assigned problems. The author did not observe any instances of peer collaboration, so is not able to describe whether this collaboration consisted of individuals comparing answers or of teams jointly solving problems. However, several teachers mentioned that when students worked together with manipulatives, the classroom did get a "little noisy" with their excitement.

4. Tools. In the comparison classrooms in this study, manipulatives were used some of the time by teachers, but the main tool was the textbook. The older the students, the less often manipulatives were used. A third-grade teacher mentioned that manipulatives were not always necessary for students, that is, they could already do the mathematics problems without the materials. Manipulatives were seen as a nice break from routine and as an aid to help students who needed extra work on a concept. One teacher said, "It's like

phonics. Not all kids need phonics but it helps everyone, just like the manipulatives. There are some kids who are OK without it, but it definitely helps everyone" (Interview 4/28/99).

5. Equity. Since instruction is delivered to whole groups, students are expected to keep up with the material. When a student experiences difficulty in learning mathematics, teachers typically inform the parents of the child and sometimes send extra practice home in the form of worksheets or flash cards. Sometimes parent volunteers may be called upon to tutor individual students during school. One teacher commented that if the whole class was having difficulty, she would go over the material again in different ways such as using manipulatives, giving explanations at the board, or pairing students in groups.

General Questions
How long have you been here at this school?
Where did you go to kindergarten?
What do you do in math class?
What materials do you use in math class?

Figure 4.4: General questions for students

General Questions During Student Interviews

Student responses to the questions about what they were learning in mathematics class and what materials were used in their classrooms (Figure 4.4) showed a difference in the two schools. Their responses agree with what the teachers said of their curriculum: Montessori students used manipulative materials to learn mathematics, and non-Montessori students used some materials but rarely in the second and third grades.

Concepts discussed by all grade levels of Montessori students include the four operations (addition, subtraction, multiplication, and division) and geometry. Operations involving the use of regrouping and numbers into the hundreds were mentioned at all grade levels. Content areas discussed by non-Montessori first graders were limited to counting in tens, single-digit addition and subtraction, and shapes. Regrouping was not mentioned until the second grade, and multiplication and division were only mentioned by the third graders. In the non-Montessori school, multiplication at the third-grade level involved times tables for single-digit numbers, but in the Montessori school, third-grade children discussed learning about multiplication of two- and three-digit numbers.

Montessori students at all grade levels described using a large variety of manipulative materials with most talking about using some type of place value materials—golden beads, stamp game, or bead frame (similar to an abacus but color coded like the stamp game)—in addition to other materials. Three third graders made a point to say they used mostly pencil and paper to do their mathematics because they were "too good" to use materials any longer. At the non-Montessori school, about half of the first graders, three of the second graders, and two of the third graders said they used materials in mathematics class; these were mostly cubes and counters. Since instruction in the non-Montessori school rarely strays from whole-class presentations, the mention of using manipulatives by even one child in a class implies the use of materials in the whole class. However, since only a few children recalled their use, the use of materials may not have been helpful or valuable for the other children.

Montessori students seemed to have a clearer idea of the purpose of their materials, and were generally able to name the concept involved. For instance, one Montessori first grader, when failing to add two three-digit numbers with pencil and paper, could however, explain the ideas of regrouping in addition when using the stamp game. (See the discussion of vertical addition problems later in this chapter for a transcript of her explanation). In contrast, one non-Montessori first grader mentioned the use of cubes in this way: “First we have a piece of paper that shows them [the cubes] put together and we have to put them [the cubes] together and count how many are there” (Interview 3/8/99). The comparison of these student responses shows a subtle difference in the use of materials in these classrooms. Children in the comparison classrooms many times used materials similarly to the way their workbook posed fill-in-the-blank type of questions: short tasks with a very specific purpose; whereas students in the Montessori classrooms were more often asked to do counting activities in which they gained familiarity with the materials. Through their familiarity, the Montessori students developed their own ideas of the value of tens and hundreds.

Analysis of Research Questions

This section will provide the data and analysis from the 93 children’s interviews. A summary of the results precedes the description of data to give the reader an advance organizer for understanding the data. A discussion of the results for each task will be presented to give the reader an idea of how the children answered each task. Statistical findings are presented alongside more general, qualitative trends in student responses. When trends were not found between school or between grade levels, the most frequent or extraordinary responses are provided from the pool of all responses. When extraordinary responses are given, they were chosen without regard to the child’s school.

Chi-squared analysis was performed on the tables of data presented, since the data are considered as nominal or ordinal data. The conventional alpha level of .05 was considered to show statistical significance. While there is disagreement over the minimum expected value frequency required to validly perform the chi squared test, this study followed the findings of several researchers (Camilli & Hopkins, 1978, 1979; Conover, 1974; Roscoe & Briars, 1971) in allowing the average expected frequencies to average a low as 2. Another restriction frequently placed upon 2x2 chi squared tests, the Yates correction, was not used, as it has been found to cause unnecessary conservative values (Camilli & Hopkins, 1978). However, the author understands that some of the expected frequencies may be considered by others to be rather low because of the small sample sizes, especially when data are analyzed by grade level.

Ten percent of the interviews were also analyzed by a doctoral student in mathematics education to verify the categorization by the author. There was 90 percent agreement on assigning categories. The author's categorizations were used in all cases.

When taken as a whole, the individual task data and further analysis shows that Montessori students consistently outperformed the non-Montessori students on tasks of a more conceptual nature, while performing the same on counting and symbolic tasks. The Montessori students were able to solve mental addition and subtraction problems using tens thinking strategies and were better able to solve application problems involving hundreds and thousands. Therefore, it is asserted that these Montessori students are still able to compute symbolically, while having a conceptual understanding of the place concepts underlying the symbolic treatment of the topic.

Subquestion A: What are the differences or similarities in the way that Montessori students complete tasks involving counting as compared to non-Montessori students in first through third grade?

Tasks involving counting, with and without materials, included the counting, unifix cubes, and strips and squares counting and uncovering tasks. Limited differences were found between the Montessori and non-Montessori students.

Counting Task Questions	Category	Sample Responses
How high can you count?		"20" "492" "2,000 and something" "6,000" "I don't know"
Would you start at 35 and count up?	*1s* (unless also successful at 10s or 100s level)	"35, 34, 33, no wait, 35, 36, 37, ..." "35, 36, 37, ..."
Would you count by 10s?	*1s* (unless also successful at 10s or 100s level)	no answer
	10s (unless also successful at 100s level)	"10, 20, 30, 40, ..."
Would you start at 60 and count by 10s?	*10s* (unless also successful at 100s level)	"60, 70, 80, 90, 100, 101, ..." "60, 70, 80, 90, 100" [stopped at 100] "60, 70, 80, 90, 100, 110, ..." [continued through to 110]
Would you count by 100s?	*10s* (unless also successful at 100s level)	"100, 101, ..."
	100s	"100, 200, 300, ..."
Would you start at 700 and count forward by 100s?	*100s*	"700, 800, 900, 1000" [stopped at 1,000] "700, 800, 900, ten hundred" [stopped at 1,000] "700, 800, 900, 1000, 1100" [continued through to 1,100]
Would you start at 24 and count backwards?		"24, 23, 22, 21, 20, ..."
Would you start at 60 and count backwards by 10s?		"60, 59, ..." "60, 50, ..."

Figure 4.5: Counting task questions, categories, and sample responses

Counting task. In this part of the interview, students were asked to count by ones, tens, and hundreds (Figure 4.5). Students' responses were coded in two ways. First, they were grouped by counting level: students who could successfully count by hundreds, starting with either 100 or with the prompt of 700, were coded as *100s*; students who declined or were unable to count by hundreds but who successfully counted by tens starting with either 10 or with the prompt of 60, were coded as *10s*; the remaining students declined or were unable to count by tens, but were able to count by ones starting with either 1 or the prompt of 35 and so were coded as *1s*. Students who could count by tens or hundreds were also classified by their rhythm of counting: whether they automatically *stopped* at 100 and 1,000 or *continued* on to 110 when counting by tens or to 1,100 when counting by hundreds. Student responses were placed in the latter category if they continued their counting sequence through either or both 100 and 1,000.

No detectable differences were found in the highest level of counting attained by students at any grade level or in the overall comparison between schools (Table 4.1). In both schools about half of the first graders, three-fourths of the second graders, and most of the third graders were able to count by hundreds.

Table 4.1

Comparison of Montessori and Non-Montessori Student Responses on Counting Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Counting responses	1st*	2nd*	3rd	1st	2nd*	3rd*
		Counting level				
1s	0	1	0	1	1	0
10s	7	1	1	11	5	0
100s	7	15	15	4	10	14
		Counting rhythm [a, b]				
stopped	9	4	3	12	10	5
continued	4	10	13	2	4	8

Grade 1 Montessori vs. non-Montessori counting level χ^2 (2, n =30) = 2.59, $p<1.00$

Grade 2 Montessori vs. non-Montessori counting level χ^2 (2, n =33) = 3.64, $p<0.20$

Grade 3 Montessori vs. non-Montessori counting level χ^2 (1, n =30) = .91, $p<1.00$

All grades Montessori vs. non-Montessori counting level χ^2 (2, N =93) = 3.53, $p<0.20$

Grade 1 Montessori vs. non-Montessori counting rhythm χ^2 (1, n =27) = 1.06, $p<1.00$

[a] Grade 2 Montessori vs. non-Montessori counting rhythm χ^2 (1, n = 28) = 5.14, $p<0.025$

Grade 3 Montessori vs. non-Montessori counting rhythm χ^2 (1, n =29) = 1.40, $p<1.00$

[b] All grades Montessori vs. non-Montessori counting rhythm χ^2 (1, n = 84) = 6.89, $p<0.01$

*Note: Six student counting rhythm responses were not able to be coded because the tapes were inaudible and interview notes did not contain counting rhythm type

When data were coded according to the students' counting rhythm, significantly more Montessori students than non-Montessori students continued their counting through 100 to 110 or through 1,000 to 1,100. This difference was significant overall and in the case of second graders (Table 4.1). Continuing to count through these numbers may imply that students have a more conceptual understanding of the ideas of counting rather than just reciting a memorized series of numbers.

An alternative explanation might be that Montessori students are able to count higher at a younger age and, therefore, have had practice reciting number sequences. However, the responses were then analyzed according to which students continued counting through both 100 and 1,000, which counted through 100 and stopped at 1,000, which stopped at 100 and counted through 1,000, and which stopped at both 100 and 1,000. These data show significant differences between third-grade student counting rhythms. Of the responses originally coded as *continued*, 83% of Montessori third graders were found to count through both 100 and 1,000 as compared to 25% of non-Montessori third graders (Table 4.2). These third-grade Montessori students seemed to have the same rhythm of counting, whether counting by tens or hundreds, while the non-Montessori third graders did not. This may imply that students in the non-Montessori school may have memorized a list of numbers to count by tens at a younger age, but may have a conceptual understanding of numbers in the hundreds after having learned to count by hundreds while understanding more about place value ideas.

Table 4.2
Comparison of Montessori and Non-Montessori Third-Grade Student Responses on Counting Rhythm (n=28)

Counting rhythm[a]	Montessori*	Non-Montessori
stopped at both 100 and 1,000	3	5
continued through 100 but stopped at 1,000	0	0
stopped at 100 but continued through 1,000	2	6
continued through both 100 and 1,000	10	2

[a] Grade 3 Montessori vs. non-Montessori counting rhythm χ^2 (2, $\underline{n}$ = 28) = 7.73, $\underline{p}$<0.025
*Note: One Montessori third grader could not count by hundreds, but when counting by tens continued through 100

Unifix Cubes Task Questions	Categories	Sample Responses
Would you guess how many are here? *(Table 4.3)*	*10-49 range*	"24"
	90-100 range	"100, they usually come in packs"
	uncodable	"30, no 50"
Would you count all these cubes for me?		child connected cubes together as she counted up to 20 and then pointed to individual cubes as she counted the rest of the numbers to 68 moved each cube as counted up to 68
How could you put them so the next child could count them even faster? *(Table 4.4)*	*don't know*	"I don't know"
	1s, 1 long line	"in a line"
	2s, 3s, 4s	"hook together twos or threes"
	5s, 6s, 8s	"groups of five"
	10s	"10 bars; 6 tens bars and 8 left over"
	other	"take away some"

Figure 4.6: Unifix cube task questions, categories, and sample responses

Unifix cubes task. When asked to guess the number of blue unifix cubes set before them (Figure 4.6), children's answers ranged from 16 to 100 when there actually were 68 cubes on the table. Data are grouped in 10-unit increments for display (Table 4.3), but were analyzed according to how many students guessed a number within 20 units of the target number of 68, how many were more than 20 units below target, and how many were more than 20 units above the target. Coding by this classification was done to see how close student estimates were to the actual number of cubes.

Differences in the estimates of the number were not statistically significant for the first or third graders, but were significant between the Montessori and non-Montessori second graders. Moreover, the Montessori second graders did the best job of guessing the number of cubes compared to all other groups. In addition to the 12 students who gave a specific answer within 20 units of 68, one student's guess was "60 to 100," another's was "50-100," and a third initially guessed 30 but changed his guess to 50 before beginning to count the cubes. Two of these students' responses were categorized as *uncodable*, while the third was scored at 30 and categorized in the *10-49 range*.

After students were asked to guess the number of cubes, they were asked to count them. Nearly all students counted by ones, although one first-grade Montessori girl made two long rows of cubes, tried to count by twos, and then stopped. Then she counted one row (32), doubled that number (64), and added it to the four remaining cubes to get 68, showing number sense and an ability to think creatively.

Table 4.3

Comparison of Montessori and Non-Montessori Student Estimates on Unifix Cubes Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Estimate	1st	2nd [a]	3rd	1st	2nd [a]	3rd
Estimate in 10-unit increments						
uncodable	1	3	1	0	1	0
10-19	2	0	0	0	0	0
20-29	0	1	1	1	2	1
30-39	1	1	4	3	1	2
40-49	1	0	0	2	3	2
50-59	1	8	2	2	3	0
60-69	2	4	3	2	1	1
70-79	1	0	2	0	0	2
80-89	1	0	1	1	2	3
90-99	0	0	0	2	2	1
100	4	0	2	3	1	2
Estimate grouped around target value of 68 [a]						
uncodable	1	3	1	0	1	0
10-49	4	2	5	6	6	5
50-89	5	12	8	5	6	6
90-100	4	0	2	5	3	3

Grade 1 Montessori vs. non-Montessori grouped around target value χ^2 (2, n = 29) = .20, p<1.00

[a] Grade 2 Montessori vs. non-Montessori grouped around target value χ^2 (2, n = 29) = 6.97, p<0.05

Grade 3 Montessori vs. non-Montessori grouped around target value χ^2 (2, n = 29) = .45, p<1.00

All grades Montessori vs. non-Montessori grouped around target value χ^2 (2, N = 87) = 4.18, p<0.20

Note: 68 cubes were on the table

Table 4.4

Comparison of Montessori and Non-Montessori Student Responses to Query Regarding Placement of Cubes to Count Faster in Unifix Cubes Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Placement choice	1st	2nd	3rd	1st	2nd	3rd
don't know	1	0	1	2	4	0
1s, 1 long line	5	2	1	4	3	0
2s, 3s, 4s	2	3	3	2	0	2
5s, 6s, 8s	1	3	1	1	2	2
10s	4	9	10	5	7	10
other	1	0	0	2	0	0

Grade 1 Montessori vs. non-Montessori χ^2 (5, $\underline{n}$ = 30) = .76, $\underline{p}$<1.00

Grade 2 Montessori vs. non-Montessori χ^2 (4, $\underline{n}$ = 33) = 7.63, $\underline{p}$<0.20

Grade 3 Montessori vs. non-Montessori χ^2 (4, $\underline{n}$ = 30) = 2.41, $\underline{p}$<1.00

All grades Montessori vs. non-Montessori χ^2 (5, $\underline{N}$ = 93) = 3.75, $\underline{p}$<1.00

Students were then asked how the cubes could be placed so that a fictitious child could count them even faster (Figure 4.6). Most children responded that the cubes should be hooked together in various fashions. Responses suggesting a child should put the cubes in one long row or hook them together as one big stick or tower were coded *1s*, because the author assumed the child would still count by ones. Responses indicating that the child should place the cubes in small groups and count by twos, threes, or fours were coded together. Similarly for those responses indicating larger groupings of five, six, or eight. Responses indicating that a child should group cubes by tens were placed into a separate category because of the importance of grouping by tens. Responses categorized as *other* included the use of a calculator, "making something" of the cubes, and "take away some." No child suggested grouping by seven or nine.

No statistically significant difference was found between Montessori and non-Montessori students at any grade level for this part of the cubes task (Table 4.4).

However, the choice of placement into tens increased from first to second to third grade in both schools, indicating that students in both schools develop the value of grouping items into tens.

Surprisingly, based upon the other results of the study, no difference was found in student responses of how to place the unifix cubes to count faster. The author assumed that Montessori students, through their extensive use of base-ten materials and more peer and teacher interaction, would have chosen to place the cubes into rows or piles of 10. However, of the 45 students who would choose to group by tens to count the cubes faster, eight students spontaneously volunteered the information that the cubes would be grouped into 6 tens and 8 ones: six Montessori students (1 first grader, 4 second graders, and 1 third grader) and two non-Montessori students (1 second grader and 1 third grader). A most interesting method came from a Montessori third-grade boy. He suggested that we put the cubes in rows of four. After a pause, he continued: "I just did some calculations to see if it would work. Four goes into 12 and 12 goes into 60 so 4 goes into 68" (Interview 1/25/99), showing another use of number sense.

Strips and Squares Counting Questions	Categories	Sample Responses
Would you put 13 squares on this paper?	*1s*	used only units to count used a strip, but counted by ones
	10s	picked up a strip and counted "10," and then counted on by ones to 13
Would you put 86 squares on this paper?	*1s*	used only units to count used strips and squares, but counted by ones
	10s	picked up strips and counted "10, 20, 30,...80" and then counted on by ones to 86

Figure 4.7: Strips and squares counting questions, categories, and sample responses

<u>Strips and squares counting task.</u> Using the felt strips and squares materials students were asked to show 13 and then 86 (Figure 4.7). Two solution methods were displayed by students: counting by *1s* (counting the requested number by single squares or by counting individual squares on the strips of 10) and counting by *10s* (using the strips to count first by tens and then using individual squares to show the remainder of the number). No statistically significant differences were found in the solution methods of the students to show 86 (Table 4.5). Two students, a first-grade non-Montessori boy and a third-grade Montessori girl, did show an ability to count by tens in the middle of a decade (e.g., 15, 25, 35, ...). Also, when the strips and squares materials were placed on the table, two Montessori second-grade boys insisted on counting all the strips and squares on the table by tens.

Table 4.5
Comparison of Montessori and Non-Montessori Student Strips and Squares Counting Task Solution Methods to Show 86 (N = 93)

	Montessori grade level			Non-Montessori grade level		
Method of counting	1st	2nd	3rd	1st	2nd	3rd
1s	4	2	0	5	1	1
10s	10	15	16	11	15	13

Grade 1 Montessori vs. non-Montessori χ^2 (1, $\underline{n}$ = 30) = .03, $\underline{p}$<1.00
Grade 2 Montessori vs. non-Montessori χ^2 (1, $\underline{n}$ = 33) = .30, $\underline{p}$<1.00
Grade 3 Montessori vs. non-Montessori χ^2 (1, $\underline{n}$ = 30) = 1,18, $\underline{p}$<1.00
All grades Montessori vs. non-Montessori χ^2 (1, $\underline{N}$ = 93) = .12, $\underline{p}$<1.00

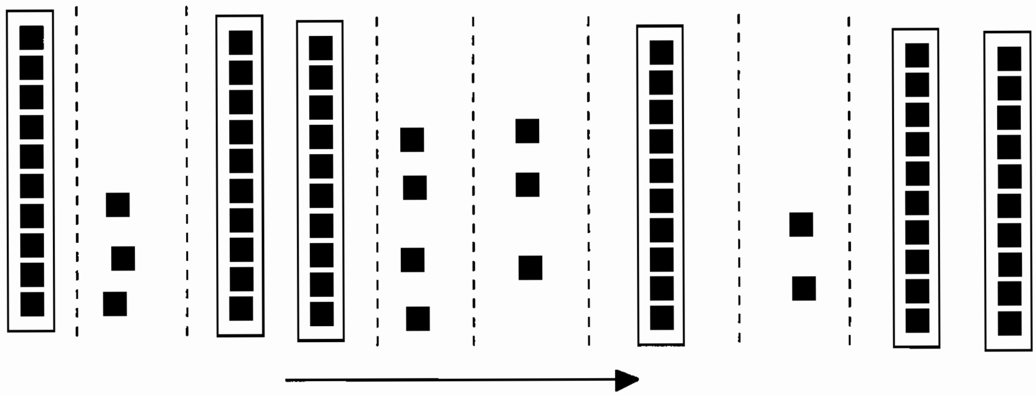


Figure 4.8: Uncovering task materials

Strips and Squares Uncovering Task	Categories	Sample Responses
"As I uncover each section of this board (Figure 4.8), tell me how many squares you see in all"	*1s*	counted each square separately from 1 to 72
	mix 10s and 1s	sometimes counted by ones and sometimes by tens (e.g. "10, 11, 12, 13, 14, 15, ...33, 34, 35, 36, 37, 40, 50, 52, 72
	reorganized	reorganized by counting the tens first by tens and then adding on the ones each time another group of squares was uncovered
	mix reorganizing and tens	reorganized the first 33 squares by tens and then counting on the rest of the time by tens and ones

Figure 4.9: Strips and squares uncovering task, categories, and sample responses

Strips and squares uncovering task. Using a board upon which many felt strips and squares were pasted, students were asked to count how many squares were seen as the board was uncovered (Figure 4.8). Student responses were categorized according to how they solved the problem (Figure 4.9): students who counted each and every square were categorized as *ones*, students who sometimes counted on by ones and sometimes counted on by tens were categorized as *mix 10s and 1s*, students who regrouped the squares on the board into tens and ones each time more of the board was uncovered were categorized as *reorganized*, and students who mixed tens and reorganizing strategies (e.g., reorganizing earlier when only 33 squares were shown, but then counting on by tens later when many more squares were showing) were categorized as *mix reorganizing and tens.* The order of the categories was considered hierarchical—using a mixed strategy of reorganizing and counting on by tens showed a more advanced use of number sense than simply reorganizing all the strips and squares or counting all of them by tens and ones. Using a ones strategy was considered to be an inferior method. No significant differences were found at any grade level or in the overall analysis between the two schools on this task (Table 4.6).

Table 4.6

Comparison of Montessori and Non-Montessori Student Methods in Counting on Strips and Squares Uncovering Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Method	1st	2nd	3rd	1st	2nd	3rd
ones	6	1	0	8	3	1
mix 10s and 1s	2	3	0	4	0	1
reorganized	0	1	1	1	3	0
mix reorganizing and tens	6	12	15	3	10	12

Grade 1 Montessori vs. non-Montessori χ^2 (3, n = 30) = 2,83, p<1.00

Grade 2 Montessori vs. non-Montessori χ^2 (3, n = 33) = 5.16, p<0.20

Grade 3 Montessori vs. non-Montessori χ^2 (3, n = 30) = 3.21, p<1.00

All grades Montessori vs. non-Montessori χ^2 (3, N = 93) = 3.08, p<1.00

Summary of Subquestion A. For the four tasks in this section, limited differences were found between the Montessori and non-Montessori students: a greater tendency by Montessori students to count past the benchmarks of 100 and 1,000 instead of stopping, better estimates for the number of unifix cubes placed on the table by Montessori second graders, and a higher rate for Montessori students of volunteering information about placing of 68 cubes into 6 groups of 10 and 8 singles as a faster way of counting the cubes. Analysis of the rest of the tasks showed that students in both schools responded similarly. These results may be related to the number and kind of counting experiences found in the two different schools. These different counting experiences may not have a large effect on the counting abilities of students, but since experience with counting affects other areas of place value, differences on later tasks may also be due to different counting experiences.

Subquestion B: What are the differences or similarities in the way that Montessori students complete tasks involving mental addition and subtraction as compared to non-Montessori students in first through third grade?

Most of the analysis for the strips and squares hidden task (involving addition and missing addend questions using materials) and the horizontal addition task (involving two-digit addition problems written on paper) showed statistically significant differences. Montessori students did better on the hidden strips and squares task and more often used tens thinking strategies rather than the standard addition algorithm when solving horizontal addition tasks.

Strips and Squares: *Hidden task*	Categories	Sample Responses
(2 strips and 3 squares visible) "I've hidden 4 squares and 1 strip; how many are there all together?"	*level 1* if correct	"37" "20, 30, ...37"
(3 strips and 8 squares visible) "I've hidden 4 squares and 1 strip; how many are there all together?"	*level 2* if correct	"fifty..., I mean forty, ...fifty...2?" "52...8 plus 4 is 12 then plus tens"
(3 strips visible) "I've hidden some under here; If there are 50 altogether, how many are covered?	*level 3* if correct	"20" "2 strips" "2 tens"
(2 strips and 4 squares visible) "I've hidden some under here; If there are 36 in all, how many are covered?"	*level 4* if correct	"1 strip and 2 squares" "12"
(2 strips and 3 squares visible) "I've hidden some under here; If there are 41 in all, how many are covered?"	*level 5* if correct	"18....42 minus 20 minus 4" "18...2 less than 20"

Figure 4.10: Strips and squares hidden task questions, categories, and sample responses

Strips and squares hidden task. Students were asked five addition and missing addend questions involving the felt strips and squares (Figure 4.10). Problems were posed in the same order for each student, but questioning ceased when a student was not able to answer two questions in a row. Attainment level in Table 4.7 was coded according to how far along in the questioning students were able to succeed. For instance, if a child could answer the first three questions correctly, but then was not able to answer the fourth or fifth question, he was coded as level *3*. If the answers to the first three questions were correct, the fourth question was almost correct, and the fifth was correct, the child would have been coded as a *5*. For instance, one child correctly answered the first three questions, answered the fourth one incorrectly (saying two squares when the solution was one strip and two squares), but answered the fifth one correctly. This child was categorized as level *5*.

Table 4.7 displays the data in two ways: by level of attainment on these five questions and then in a collapsed form. Data were collapsed into three categories: (a) *none*; (b) *1/2/3*, which required addition or a subtraction only involving tens; and (c) *4/5*, more difficult questions involving subtraction with and without regrouping with tens and ones. Statistically significant differences were found between students in grade 3 and overall, showing an ability of the Montessori students to solve unfamiliar problems using tens and ones. Interestingly, the second grade non-Montessori students did slightly better than the third graders at the same school. These second graders may not have been as familiar with the standard addition and subtraction algorithms and possibly were more open to other methods of solving problems, thus allowing them to be more successful.

Table 4.7

Comparison of Montessori and Non-Montessori Student Responses on Hidden Strips Task Questions (N = 93)

	Montessori grade level			Non-Montessori grade level		
Response	1st	2nd	3rd	1st	2nd	3rd
Highest level of correct response						
none	1	0	0	3	2	0
1	1	0	0	2	0	0
2	2	2	1	4	0	2
3	2	4	0	3	2	5
4	4	3	4	2	5	1
5	4	8	11	2	7	6
Highest level of correct response: Collapsed categories [a, b]						
none	1	0	0	3	2	0
1/2/3	5	6	1	9	2	7
4/5	8	11	15	4	12	7

Grade 1 Montessori vs. non-Montessori when categories were collapsed χ^2 (2, n = 30) = 3.36, p<0.20

Grade 2 Montessori vs. non-Montessori when categories were collapsed χ^2 (2, n = 33) = 4.02, p<0.20

[a] Grade 3 Montessori vs. non-Montessori when categories were collapsed χ^2 (1, n = 30) = 7.31, p<0.01

[b] All grades Montessori vs. non-Montessori when categories were collapsed χ^2 (2, N = 93) = 5.99, p<0.05

The hidden strips task responses were also analyzed by the way in which students solved the problems. For instance, one child solved the first problem (14 hidden and 23 showing; how many in all?) by starting at 14 and adding 23 more by ones (counting on) to get the correct answer of 37. This child was coded as *1s*. Another child added the three strips to get 30 and then added the 7 ones to get 37. This child was coded as using a *10s/1s* strategy. No statistically significant differences were found for student solution methods (Table 4.8). However, many nice solution methods were found by students at all grade levels and in both schools, especially for the last and most difficult question (a missing addend problem involving regrouping). This question involved asking students how many strips and squares were hidden under a cloth if 23 were showing and there were 41 in all. The answer was 18. Many students (27) started their answer saying two strips were hidden, but changed their mind when they realized that would be too many. One Montessori first grade girl first said 27, but then realized that the total would be too large and changed her answer to 18. When asked how she solved the problem, she said "since I was wrong by 9, I just counted back 9" (Interview 3/17/99). Two other Montessori girls similarly said they counted back 2 from 20 when they realized that two strips hidden would make the total too large. A non-Montessori first grade boy took 41 and subtracted 20 and then subtracted 3 more to get 18.

Table 4.8

Comparison of Montessori and Non-Montessori Student Method of Solution on Hidden Strips Task Questions (N=93)

	Montessori grade level			Non-Montessori grade level		
Method of solution	1st	2nd	3rd	1st	2nd	3rd
ones	6	0	0	10	2	1
tens	8	17	16	6	14	13

Grade 1 Montessori vs. non-Montessori χ^2 (1, n = 30) = 1.16, p<1.00
Grade 2 Montessori vs. non-Montessori χ^2 (1, n = 33) = 2.26, p<0.20
Grade 3 Montessori vs. non-Montessori χ^2 (1, n = 30) = 1,18, p<1.00
All grades Montessori vs. non-Montessori χ^2 (1, N = 93) = 3.43, p<0.10

16 + 9 =
28 + 13 =
38 + 24 =
39 + 53 =

Figure 4.11: Horizontal addition task questions

Horizontal Addition Task	Categories	Sample Responses
"Without paper, would you add these numbers?" (see Figure 4.11)	*1s*	16 + 9: "16, 17, 18, 19, 20, 21, 22, 23, 24, 25" using fingers to count on by ones
	10s/1s	28 + 13: "thirty, no, forty one" 38 + 24: "62" (an immediate response) 38 + 24: "38 plus 20 equals 58, plus 4 equals 62" 39 + 53: "eighty, ninety two"
	algorithm	uses the term "carrying" when asked how the problem was solved 28 + 13: "41"; "put this under there" (uses finger to put 13 under the 28) 39 + 53: "it would be fifty, no, yeah, yeah, 54"; how did you get that; "I added these two together [8 and 4] and got 14 and carried the 1"

Figure 4.12: Horizontal addition task, categories, and sample responses

Horizontal addition task. A continuing review of the literature uncovered this question that was added after half of the interviews had been completed (Figure 4.11). Therefore, only approximately half of the subjects were asked this question. Students were categorized into four groups: those who could not solve the problem, those who used a strategy of counting on by ones (*1s*), those who used ideas of tens and ones to add (*10s/1s*), and those who used the standard addition algorithm (*algorithm*) (Figure 4.12). The analysis revealed significant differences for grade 3 and overall showing that

Montessori students used the concept of tens to add the numbers while non-Montessori students used the standard addition algorithm (Table 4.9).

Students using a *ones* strategy solved "16 + 9" by starting at 16 and counting on 9 more by ones using their fingers. Students who used the idea of tens to solve the problem used various methods. The fastest solutions were those given by students who merely said that the answer to 39 plus 53 would be "eighty....no, ninety two" (Interview 3/31/99). Other solutions include a student adding 38 plus 24 to first get "38 plus 20 equals 58, plus 4 equals 62" (Interview 3/31/99). Another student solved 28 plus 13 by saying "I got my 11 and then I added my two, I mean three tens together" (Interview 3/19/99).

Students who used the standard algorithm showed this by using their finger to trace one number under another and mimicked the *carrying* by moving their finger above the ten's place. Some mentioned the word *carrying*.

As mentioned in chapter 3, Cobb and Wheatley (1988) found that children who used their own invented thinking strategies for horizontal addition were frequently more successful at solving these problems correctly even though they used the standard algorithm for vertical addition, sometimes getting the answer wrong. In this study many students who solved the problems using the standard algorithm did so incorrectly (19% incorrect responses using *algorithm* method vs. 4% incorrect responses using *10s/1s* method) because of the many numbers to remember all at once (e.g., the visualized number to be added beneath the first number, the first addition in the units place, the carrying of the ten's place, and the answer to the tens digit addition).

Table 4.9

Comparison of Montessori and Non-Montessori Student Responses on Horizontal Addition Task Questions (n=54)

Solution method	Montessori grade level[c] 1st	2nd[a]	3rd[b]	Non-Montessori grade level[c] 1st	2nd[a]	3rd[b]
none	1	2	0	5	0	0
ones	4	0	1	5	1	1
algorithm	0	0	1	0	4	8
tens	4	7	5	2	3	0

Grade 1 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 21) = 3.08, $\underline{p}$<1.00

[a] Grade 2 Montessori vs. non-Montessori χ^2 (3, $\underline{n}$ = 17) = 8.57, $\underline{p}$<0.05

[b] Grade 3 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 16) = 10.36, $\underline{p}$<0.01

[c] All grades Montessori vs. non-Montessori χ^2 (3, $\underline{n}$ = 54) = 15.69, $\underline{p}$<0.01

When the data for the horizontal addition task is analyzed by grade level and school, other differences in the schools are revealed. For interviews conducted in the non-Montessori school, 42% of the first graders could not answer the questions, another 42% solved by counting on by ones, and the remainder solved by using tens strategies. In the second grade, solution methods were almost evenly split between use of the standard addition algorithm and tens strategies. By the third grade, however, most students solved the problem using the standard addition algorithm. This movement away from tens strategies and toward the use of the standard algorithm for the older children is striking since, for this problem, using the standard algorithm was awkward and sometimes resulted in incorrect answers. For the Montessori interviews, a different pattern is revealed. Strategies of the first graders at this school were evenly split between counting on by ones and using tens strategies. Most of the second and third graders used tens strategies, even though these students knew how to perform the standard addition algorithm by second grade (as evidenced by data from the vertical addition task not yet discussed).

The overwhelming use of tens strategies by these Montessori students may imply that these students have a deeper understanding of two-digit numbers and the operation of addition or at least that they have a greater facility with such numbers. The non-Montessori students did not show such facility with these numbers and seemed to be tied to the use of the standard addition algorithm.

Summary of Subquestion B. Although these two tasks (hidden strips and horizontal addition tasks) were similar in nature with regard to the underlying necessity to add tens and ones in two-digit tasks, analysis of solution methods showed different results. When students were asked questions involving materials, no difference was found in solution methods between the schools. However, when the questions were posed on paper as horizontal addition tasks, Montessori students tended to continue their use of ten's and ones strategies while non-Montessori students (especially at the third-grade level) switched to using the standard addition algorithm.

The Montessori students were able to use tens thinking strategies for both tasks, possibly showing that their extensive use of materials have not made them dependent upon materials; rather, they are able to use the same strategy even in the absence of materials. However, the presence of base-ten materials may have supported the non-Montessori students' thinking strategies; while the absence of materials may have required them to rely on the method used most often in their classrooms: the standard addition algorithm. This mismatch of strategies occurred even though the horizontal addition questions were asked within five minutes after the hidden strips task.

Subquestion C: What are the differences or similarities in the way that Montessori students complete tasks involving the forming and unpacking of hundreds and thousands as compared to non-Montessori students in first through third grade?

Analysis of the flower and bean tasks revealed statistically significant differences favoring Montessori students. Since these tasks were novel to both the Montessori and non-Montessori students, these results illustrate consistently higher levels of generalizing place value concepts into the hundreds and thousands for the Montessori students.

Flower task questions	Category	Sample Responses
"What if you went outside to pick some flowers for (principal's name). Each flower has 10 petals. You found 2 loose petals on the ground and picked 10 whole flowers. How many petals would you have?"	*100s*	"100, 2 fell off, 98" (used fingers to count by tens to 100 then subtracted 2 "102"; why?; "10 times 10 is 100 and 2 more" "100"
If child is unable to answer correctly, she is asked about 3 loose petals and 4 whole flowers.	*10s*	"10+10 is 20 so you add 2 then I went 30 and then 40 and 3 on the ground is 43" "43"
	none	child unable to answer either question

Figure 4.13: Flower task questions, categories, and sample responses

Flower task. During the flower task, a child was first asked how many petals there were altogether on 10 flowers with 10 petals each including 2 loose petals (Figure 4.13). Several different responses were counted as correct and coded *100s question*: (1) 102 petals, (2) 98 petals (as if the child believed that 2 of the petals fell to the ground from the

100 total petals), and (3) 100 petals (as if the child believed that 2 fallen petals plus 98 remaining petals were combined). Most children answering this question correctly answered 102. After giving an incorrect response to the 100s question, a child would be asked about 3 loose petals and 4 whole flowers. The same latitude in response was given for this question, and this question was coded *10s* question if correct.

Table 4.10
Comparison of Montessori and Non-Montessori Student Responses on Flower Task (N = 93)

	Montessori grade level[b]			Non-Montessori grade level[b]		
Level of answer	1st[a]	2nd	3rd	1st[a]	2nd	3rd
none	3	2	1	10	5	2
10s question	1	4	0	4	2	1
100s question	10	11	15	2	9	11

[a] Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 10.82, p<0.01
Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 2.12, p<1.00
Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 1.82, p<1.00
[b] All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 8.96, p<0.025

The analysis of children's answers to the flower task revealed statistically significant differences between first graders and overall (Table 4.10). Overall, 79% of Montessori and 48% of non-Montessori students answered the question correctly. Of the first graders, the percentages were 71 and 13, respectively. No statistically significant differences were found for second and third graders, who have probably worked with numbers up to 100 in their respective schools, and may therefore be expected to solve a problem such as this. The difference in the first graders' responses may reflect the learning of numbers through 100 by Montessori students in the first grade or before, earlier than the non-Montessori students learn these concepts.

When students were asked to explain their solutions, however, more first and second grade non-Montessori students used a strategy of counting by tens to get to 100 whereas more students from the Montessori school used a strategy of multiplying 10 by 10 to get to 100. The latter strategy was quicker, more direct, and shows an understanding of multiplication ideas.

Bean task questions	Category	Sample Responses
"If you had beans like these, and you put beans in rows of 10, how many full rows of 10 beans could you make from 35 beans?"	-	"3" "30" {incorrect}
"from 72 beans?"	*10s*	"7" "7 rows and 2 in a pile" "70" (incorrect, coded as none)
"from 123 beans?"	*almost 100s* (categorized as *10s* in for this analysis)	"10"
	100s	"12" "10 and 2 more rows is 12"
"If you put 100 beans in a cup, how many cups could you fill from 423 beans?"	-	"4" "4 and three rows of ten"
"from 1,214 beans?"	*almost 1000s* (categorized as *100s* for this analysis	"10" "11"
	1000s	"12"; why? "10 hundreds in 1000 and 2 more hundreds"

Figure 4.14: Bean task questions, categories, and sample responses

Bean task. Questioning on this task proceeded from easier questions to more difficult ones (Figure 4.14). Credit for the *10s* category was given for a correct answer to how many rows of ten could be formed from 72 beans (7 rows). Credit for the *100s* category was given for students giving an answer for the number of rows of 10 formed from 123 beans (12 rows). Credit for the *1000s* category was given for an answer for the number of cups of 100 filled from 1,214 beans (12 cups). The author considered giving credit for an answer of 10 for the last two categories (*almost 100s* and *almost 1000s*), but analysis revealed no difference in the level of significance when this was done.

Table 4.11

Comparison of Montessori and Non-Montessori Student Responses on Bean Task (N = 93)

	Montessori grade level[a]			Non-Montessori grade level[a]		
Level of answer	1st	2nd	3rd	1st	2nd	3rd
none	4	4	0	11	3	0
10s	4	2	3	4	6	6
100s	2	4	6	1	4	3
1000s	4	7	7	0	3	5

Grade 1 Montessori vs. non-Montessori χ^2 (3, n = 30) = 7.5, p<0.10

Grade 2 Montessori vs. non-Montessori χ^2 (3, n = 33) = 3.72, p<1.00

Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 2.21, p<1.00

[a] All grades Montessori vs. non-Montessori χ^2 (3, N = 93) = 8.23, p<0.05

In the bean task, as in the flower task, Montessori students were able to answer at a higher level than students in the non-Montessori school (Table 4.11). Of the Montessori students, 38% were able to answer the thousands question as compared to 17% of the non-Montessori students, indicating a higher level of place value understanding.

Some students who were able to solve the tens problem (how many rows of ten in 72) counted by tens or fives, but most answered immediately, showing a grasp of place value understanding for tens. Most students solved the more difficult hundred and thousand problems (how many rows of ten in 123 and how many cups of 100 in 1,214) in a similar manner to a non-Montessori third grader who said "in a thousand there's ten hundred plus two hundred more" (3/8/99). Even three Montessori first graders were able to express their method in this manner, which was surprising to the author even though Montessori first graders regularly work with thousands materials in mathematics. One Montessori first grader, when asked for the number of rows of ten in 123 counted by tens, putting up one finger for each count of ten. When she had put up all her fingers she looked confused and started over, the author offered her the use of the author's fingers, and she was able to count by tens to 120 and answered 12 for the number of rows of ten.

Summary of Subquestion C. The results of both the flower and bean problems show statistically significant differences favoring the Montessori students. These results may indicate a greater ability by these Montessori students to function at higher levels of the Dienes' framework. The third level of the framework requires students to be able to understand the equivalence of 10 ones for a ten, 10 tens for a hundred, and 10 hundreds for a thousand. The flower and bean tasks assess these concepts and show the Montessori students have a better ability to solve these types of problems.

Subquestion D: What are the differences or similarities in the way that Montessori students complete symbolic tasks as compared to non-Montessori students in first through third grade?

No statistically significant differences were found in the three questions involving symbolic tasks: number cards, digit cards, and vertical addition. Montessori and non-Montessori students responded to these tasks in a similar manner.

Number cards tasks	Categories	Sample responses
Showing card with 1,247 and 1,326: "Which is bigger?" "Why?"	*1000s*	"1,326"
Showing card with 358 and 372: "Which is bigger?" "Why?"	*100s* (unless student can answer the thousands level correctly)	"372"
	none	incorrect responses for other levels

Figure 4.15: Number cards questions, categories, and sample responses

Number cards task. Responses indicating that 1,326 was larger than 1,247 (Figure 4.11) were coded as *1000s* in this task. Those who could not answer this question, but were able to answer the easier questions (that 372 was larger than 358), were coded *100s*. Those students not able to answer either question were coded none. No statistically significant differences were found between student responses at the two schools (Table 4.12).

Table 4.12
Comparison of Montessori and Non-Montessori Student Responses on Number Cards Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Level of answer	1st	2nd	3rd	1st	2nd	3rd
none	4	0	0	6	2	0
hundreds	2	1	0	4	2	0
thousands	8	16	16	6	12	14

Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 1.22, p<1.00
Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 2.88, p<1.00
All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 3.21, p<1.00

Students who correctly answered the first number cards question were generally able to express their reasoning, stating that 1,326 was larger than 1,247 because the 3 is larger than the 2 or because 300 is larger than 200. To compare 1,326 and 1,247, a second-grade Montessori girl said, "3 is bigger than 2 [for the hundreds digits] and 4 is bigger than 2 [for the tens digits], but the whole point that it makes it bigger is that it has the biggest number at the beginning" (Interview 2/10/99). A third-grade Montessori girl said, "you count to 200 before you count to 300" (Interview 1/22/99), showing an understanding of large numbers in terms of their position on the number line, rather than just in the comparison of digits. A similar reasoning was found from a second grade non-Montessori student who explained why 372 was larger than 358 in this way, "58 is close to 60 and 72 is bigger than 60" (Interview 3/5/99).

Digit cards task Show cards with digits written on them: 1, 3, 4, 5. Show that 1 next to the 3 makes a 13. "What is the biggest number you can make with these cards? Can you make an even bigger number?" Sometimes only the digits 1, 3, and 5 were given.

Figure 4.16: Digit card task questions

Digit cards task. Because this question (Figure 4.16) occurred near the end of the interviews, some students were not asked this question and others were rushed. Also, based on answers to earlier questions, some students were asked to show the largest number to be made from three digits (1, 3, and 4), while others were asked the question using four digits (1, 3, 4, and 5). Therefore, only general impressions of student answers will be given.

There seemed to be no differences in the ability of students across schools. Some students were creative in their answers; when students were urged to find a larger number even when they had correctly answered the problem, six students (four non-Montessori and two Montessori students) put spaces in between the digit cards to represent zero, getting answers such as 15,043 or 10,543. When students were asked what would be necessary to make an even larger number, some replied that they would need another digit, usually saying a 6.

When students were asked to justify their answers, most explained by saying that the 5 was larger than the 4, and so on. Two Montessori third graders volunteered the information that the 5 would be placed to the left of the 4 because that would make a number in the 5,000s while placing the 4 at the left position would make a number only in the 4,000s.

Vertical addition task	Categories	Sample Responses
13 + 48	*level 1* if correct	"61" "51"
125 + 492	*level 2* if correct	"can't do this" "617"
2,474 + 5,823	*level 3* if correct	"8,297"
68,183 + 27,207	*level 4* if correct	"95,390"
134,223 + 245,912	*level 5* if correct	"370,135" "380,135"

Figure 4.17: Vertical addition task questions, categories, and sample responses

Vertical addition task. During the final question, students were allowed to use a pencil (Figure 4.17). Coding was based upon the highest level at which students thought they could solve the problems (level of guess) and the highest level at which they correctly solved the problems (level of answer). After asking which problem a child thought he could solve, the author asked the child to solve that problem. If the answer was incorrect, the interviewer asked previous questions until the child could solve the problem. If the child answered correctly the first problem asked by the author, then the author proceeded down the list until the child could not solve a problem. For instance, if a child believed that he could solve the third problem, but could not do so correctly, he was asked to solve the second. If he solved this correctly, his response was coded 2. If he could not solve the second problem, he was asked to solve earlier problems. If a child believed she could solve the fourth problem and did so correctly, she was then asked to solve the fifth problem. If she could solve the fifth, she was coded *5* and, if not, she was coded *4*. No differences were found for students' actual level of adding the numbers nor for their initial guess of their ability to solve the problems (Table 4.13).

Table 4.13
Comparison of Montessori and Non-Montessori Student Responses on Vertical Addition Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Response	1st	2nd	3rd	1st	2nd	3rd
			Level of guess			
none	1	0	0	7	0	0
1st	2	3	1	4	3	2
2nd	6	0	1	4	5	0
3rd	0	4	6	0	1	2
4th	2	0	1	1	1	1
5th	3	10	7	0	6	9
			Level of answer			
none	6	3	2	10	1	1
1st	6	2	0	5	1	1
2nd	0	0	0	1	0	0
3rd	0	1	1	0	2	0
4th	0	0	1	0	0	0
5th	2	11	12	0	12	12

Grade 1 Montessori vs. non-Montessori level of guess χ^2 (4, n = 30) = 8.81, $p<0.10$

Grade 2 Montessori vs. non-Montessori level of guess χ^2 (4, n = 33) = 8.78, $p<0.10$

Grade 3 Montessori vs. non-Montessori level of guess χ^2 (4, n = 30) = 3.47, $p<1.00$

All grades Montessori vs. non-Montessori level of guess χ^2 (5, N = 93) = 9.82, $p<0.10$

Grade 1 Montessori vs. non-Montessori level of answer χ^2 (3, n = 30) = 3.98, $p<1.00$

Grade 2 Montessori vs. non-Montessori level of answer χ^2 (3, n = 33) = 1.68, $p<1.00$

Grade 3 Montessori vs. non-Montessori level of answer χ^2 (4, n = 30) = 3.21, $p<1.00$

All grades Montessori vs. non-Montessori level of answer χ^2 (5, N = 93) = 2.12, $p<1.00$

When one non-Montessori first grader was solving the second problem (125 + 492) and was asked what the "1" was, he explained, "that means eleven so you put 1" (Interview 1/15/99). Actually the 1 in this problem would be a 100 not a 10, but this student was the only student to talk specifically of renaming numbers. Another child in the same class did not know the standard addition algorithm, but added 48 plus 13 by starting at 48 and counting 13 more on her fingers. When asked why she did not start with 13, she said "you have to add on 48 and that's harder" (Interview 1/15/99).

Three first grade Montessori students claimed that they could not do the problems on paper, but could do them with materials; one of them and two second graders could explain what to do. One explanation for 125 + 492 with the Montessori materials proceeded like this:

> Student: I use the stamp game to do it...first we would set out 1 hundred ones and put 4 hundred ones; then we take our tens and we put 2 tens beside our 1 [hundred] and 9 beside our 4 [hundred] and we put 5 [units] beside our 2 [tens] and 2 [units] beside our nine [tens].
> Interviewer: Then what?
> Student: Then we put these two together [hundreds] and these two together [tens] and these two together [units] and then we count these two and count these two and count these two and then we add them together and we put the one we came up with [units added together] there.
> Interviewer: What happens to the 2 tens and 9 tens?
> Student: We put two up here and circle it and we take away how many it was over ten.
> Interviewer: What do you do with the ten tens?
> Student: We put it back in the box.
> Interviewer: You just get rid of it?
> Student: We used to replace it with a hundred. (Interview 3/1/99)

Because of earlier responses by this child and the fact that no materials were present for the child to manipulate, the author assumed the child understands more than a rote manipulation of materials. This response showed that, even though this first-grade girl could not yet solve the problem symbolically, she was able to understand and *explain* the process of adding three-digit numbers. By the time this child enters third grade, she will probably be able to solve problems such as these without the use of materials, similar to the rest of the third graders in the study. However, this child will be able to do the computation as well as understand the ideas of place value, which will help her to understand and compute multiplication and division problems with multidigit numbers.

A Montessori third grader, on the way back to his classroom after the interview, was not impressed by his ability to do all of the interview tasks at the highest level and said, "it doesn't matter how big the number is, I can add it...it's just like adding lots of units" (Interview 2/5/99). The author assumed the boy meant units of ones, tens, and hundreds, showing his generalization of the concept of addition with multidigit numbers.

Summary of Subquestion D. In the analysis for the tasks in this section, no statistically significant differences were found. This is predictable, but possibly for different reasons in the two schools. Better comprehension of the place value ideas for Montessori students would lead to better competence at tasks involving application of ideas. The non-Montessori classrooms emphasize symbolic manipulation and so it is no surprise that these students would fare well on symbolic types of tasks.

A summary of the results of the data analysis so far (Figure 4.18) is given to help the reader review at a glance the findings described so far. When taken as a whole, the data show that Montessori students consistently outperformed the non-Montessori students on tasks of a more conceptual nature, while performing the same or slightly better on counting and symbolic tasks.

One explanation for this pattern may simply be that Montessori students begin using base-ten materials while they are in preschool while non-Montessori students may not. This may cause the students to have more experience with tens, hundreds, and thousands at a younger age. For instance, one of the Montessori teachers mentioned that she expects most of her beginning first graders to add four-digit numbers without regrouping as long as they use materials. It is possible that the better performance of Montessori students results from this earlier start to their learning about place value. If so, we might assume that the non-Montessori students would eventually perform at the same level as the Montessori students, but just a little later in their schooling. However, this seems not to be the case. When looking at the responses on the horizontal addition task, the Montessori and non-Montessori students show a qualitative difference in their approach to the problem: using tens strategies versus using the standard addition algorithm. Also, since many of the non-Montessori students were able to respond to the hidden strips and squares task using tens thinking strategies, it seems that the use of materials did help support their thinking. Unfortunately, when the materials were not present, this strategy seemed unavailable to them.

Task group	Task	Statistically significant results	Trends in responses	Non-significant results
Counting	Counting	counting rhythm (Mont>non-Mont for grade 2 and overall) (Fig. 4.2)		counting level (Fig. 4.1)
	Unifix cubes	estimate of number of cubes (Mont closer than non-Mont for grade 2) (Fig. 4.3)	responded that 68 cubes could be placed into 6 groups of 10 and 8 singles (Mont 43% vs. non-Mont 4%)	estimates (overall) placement of cubes to count faster (Fig. 4.4)
	Strips and squares counting			method of counting to 86 (Fig. 4.5)
	Strips and squares uncovering			method of counting strips and squares on uncovering board (Fig. 4.6)
Mental addition and subtraction	Strips and squares hidden	level of response (Mont>non-Mont for grade 3 and overall) (Fig. 4.7)		solution methods (Fig. 4.7)
	Horizontal addition	solution methods (Mont using tens strategies while non-Mont using algorithm in grade 3 and overall) (Fig. 4.9)		

Figure 4.18: Comparison of statistically significant and non-significant results between Montessori and non-Montessori student responses on all tasks

Task group	Task	Statistically significant results	Trends in responses	Non-significant results
Forming and unpacking 100 and 1,000	Flower task	level of response (Mont>non-Mont for grade 1 and overall) (Fig. 4.10)	more first- and second-grade Mont than non-Mont students solved hundreds task by multiplying 10x10 instead of counting by tens	
	Bean task	level of response (Mont>non-Mont overall) (Fig. 4.11)		
Symbolic	Number cards			level of answer (Fig. 4.12)
	Digit cards			responses
	Vertical addition			level of estimate level of correct solution (Fig. 4.13)

Figure 4.18 (cont.): Comparison of statistically significant and non-significant results between Montessori and non-Montessori student responses on all tasks (cont.)

Overall research question: What are the differences or similarities in the way that Montessori students respond to tasks involving place value concepts as compared to non-Montessori students in first through third grade?

The final data analysis provided in this section attempts to look at the data from a more general view. In the previous analysis, data was analyzed according to individual tasks. The analysis done in this section combined tasks according to the type of questions asked of the students. Tasks were arranged in four groupings (see Figure 4.18): two conceptual task groupings (mental addition/subtraction and forming/unpacking hundreds and thousands) and two procedural task groupings (counting and symbolic tasks). In the procedural tasks students were able to successfully complete the task using knowledge of a procedural nature. For instance in the counting tasks, students needed rote knowledge of counting and in the vertical addition tasks, students needed the knowledge of the standard addition algorithm. Conceptual tasks required students to apply information in new ways. For instance in hidden strips and squares tasks, students used their knowledge of counting, addition, and subtraction to solve the addition and missing addend questions. However, success at this task required more than rote application of the ideas, but an ability to count and add in a novel situation. Of course, some students used knowledge of a conceptual nature in the *procedural* tasks as well as using procedural knowledge in *conceptual* tasks.

Data analysis proceeded as follows. Scores were assigned based upon the categorization done in the previous sections. Percentages for each of the four task groupings were found and break points for levels of high, medium, and low responses were determined. These break points were found by looking at the data to find natural breaks in the data which would separate the data into three nearly even groups (i.e. approximately one third of the data were categorized as low, one third as medium, and one third as high). Chi squared analysis was performed for each task group. In the final stages, percentages for the two conceptual task groupings were combined for a conceptual score, and similarly, a procedural score was found. Again, break points for levels of high, medium, and low responses were determined.

A word of caution in interpreting this section. The numbers assigned to each task and to overall values in the conceptual and procedural scores were determined by using ordinal data. Therefore, the Chi squared test continued to be used and not a different test which assumed continuity of values. Therefore, the graphs found in Appendix D should be treated as a helpful pictorial representation of data but cannot be used to state conclusions about the two schools.

Counting task group analysis. The counting task grouping is the first of two procedural groupings. Six scores were added to find a combined score for this task grouping (Figure 4.19). The analysis of counting rhythm done in a previous section could not be used in this score, since all students were not categorized for various reasons. A score of 14 on counting was possible. Percentages ranged from 8% to 100% and were converted to a category of high, medium, or low (Figure 4.20). Table 4.14 shows the breakdown.

No statistically significant differences were found for students on counting tasks. Montessori and non-Montessori students both showed an expected trend of higher levels for older students. This result is in agreement with previous results for the individual tasks contained in the counting task grouping.

Task	Category	Score
Counting (see Figure 4.5) Would you count by tens? Would you count by hundreds?	*1s* *10s* *100s*	0 2 4
Unifix Cubes--counting (see Figure 4.6) Would you count all these cubes?	*1s* *10s*	0 2
Unifix Cubes--next person (see Figure 4.6) How could you put these cubes so the next person could count them even faster?	*other* *10s*	0 2
Strips and Squares Counting Method (see Figure 4.7) Would you put 13 squares on this paper?	*1s* *10s*	0 1
Would you put 86 squares on this paper?	*1s* *10s*	1 2
Strips and Squares Uncovering (see Figure 4.9) I'll uncover the squares on this board and you tell me how many you see altogether each time.	*1s* *mix 10s/1s* *reorganized* *mix reorganizing and 10s*	0 1 2 3

Figure 4.19: Counting task group questions, categories, and scoring

8	33	58	66	75	83
16	41	58	66	75	83
16	41	66	66	83	83
16	41	66	66	83	83
16	41	66	66	83	83
16	41	66	66	83	83
16	41	66	66	83	83
25	41	66	66	83	83
25	50	66	66	83	83
33	50	66	66	83	83
33	50	66	66	83	83
33	50	66	66	83	83
33	50	66	66	83	83
33	50	66	75	83	83
33	58	66	75	83	100
	58		75		100
		low	0-62		
		med	63-79		
		high	80-100		

Figure 4.20: Percentages and break points for counting group tasks

Table 4.14
Comparison of Montessori and Non-Montessori Student Response Levels on Counting Group Tasks (N = 93)

	Montessori grade level			Non-Montessori grade level		
Category	1st	2nd	3rd	1st	2nd	3rd
low	8	3	1	13	7	1
med	5	8	6	2	4	6
high	1	6	9	1	5	7

Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 2.35, p<1.00
Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 2.99, p<1.00
Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 0.12, p<1.00
All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 4,34, p<0.20

Symbolic task group analysis. The symbolic task grouping is the second of the two procedural groupings. Two scores, for the number cards and vertical addition tasks, were added to find a combined score for this task grouping (Figure 4.21). Data from the digit cards task could not be used because not all students were asked this question. A score of 17 on symbolic tasks was possible. Percentages were converted to a category of high, medium, or low (Figure 4.22). Table 4.15 shows the breakdown of levels.

No statistically significant differences were found for students on the symbolic task grouping (Table 4.15). This result is in agreement with previous results for the individual tasks contained in the counting task grouping.

Task	Category	Score
Number Cards (see Figure 4.15) I'll show you two numbers and I want you to tell me which one is bigger and why.	*none* *100s* *1000s*	0 3 7
Vertical Addition (see Figure 4.17) What level of problem can be done correctly	*0* *1* *2* *3* *4* *5*	0 2 4 6 8 10

Figure 4.21: Symbolic task group questions, categories, and scoring

0	35	52	100	100	100
0	41	52	100	100	100
0	41	52	100	100	100
0	41	52	100	100	100
0	41	52	100	100	100
0	41	52	100	100	100
0	41	52	100	100	100
11	41	52	100	100	100
11	41	52	100	100	100
11	41	52	100	100	100
17	41	58	100	100	100
17	41	76	100	100	100
17	52	76	100	100	100
17	52	76	100	100	100
17	52	76	100	100	100
29		88		100	
		low	0-45		
		med	46-81		
		high	82-100		

Figure 4.22: Percentages and break points for symbolic group tasks

Table 4.15

Comparison of Montessori and Non-Montessori Student Response Levels on Symbolic Group Tasks (N = 93)

	Montessori grade level			Non-Montessori grade level		
Category	1st	2nd	3rd	1st	2nd	3rd
low	9	3	2	11	2	1
med	3	3	2	5	5	1
high	2	11	12	0	9	12

Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 2.58, p<1.00

Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 0.87, p<1.00

Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 0.54, p<1.00

All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 0.81, p<1.00

Procedural tasks analysis. When scores from the two procedural task groupings for each student were added, scores ranged from 8 to 200 (Figure 4.23). A response level of low, medium, or high was assigned to each student's score. Analysis of these categories (Table 4.16) showed no statistically significant differences in the performance of Montessori and non-Montessori students on procedural tasks at any grade level nor overall. Older students in both schools are more competent at these tasks than younger students, which is to be expected. Therefore, Montessori students, who focus their mathematics work on the use of manipulative materials without initial explicit instruction on procedural knowledge, are not hindered in their procedural task abilities.

8	74	116	147	166	183
16	74	118	150	166	183
16	75	118	150	166	183
33	82	118	154	175	183
33	83	118	158	175	183
41	85	118	159	175	183
41	85	118	166	175	183
42	91	124	166	183	183
44	93	124	166	183	183
50	95	124	166	183	183
52	107	126	166	183	183
58	107	127	166	183	183
61	107	133	166	183	183
66	107	134	166	183	200
68	110	141	166	183	200
68		142		183	
		low	0-112		
		med	113-170		
		high	171-200		

Figure 4.23: Scores and break points for procedural tasks

Table 4.16

Comparison of Montessori and Non-Montessori Student Response Levels on Procedural Tasks (N=93)

	Montessori grade level			Non-Montessori grade level		
Category	1st	2nd	3rd	1st	2nd	3rd
low	9	3	0	14	3	2
med	4	8	8	2	8	4
high	1	6	8	0	5	8

Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 2.63, p<1.00

Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 0.06, p<1.00

Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 3.21, p<1.00

All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 2.77, p<1.00

Hidden task analysis. The first group of conceptual tasks included the hidden strips and squares task and the horizontal addition task. Unfortunately since the horizontal addition task was added after half of the interviews had taken place, data is not available for all students on this task. The author decided not to use this task in this analysis. Therefore, only the hidden strips and squares task is analyzed in this group. However, this task has already been analyzed (Table 4.7) by finding the Chi squared statistic for the categories seen in Figure 4.24. No statistically significant results were found until the data were grouped into three categories: those students not able to answer, those in categories *1/2/3*, and those in categories *4/5*.

The analysis in this section proceeded by giving scores to each category (Figure 4.24), finding breakpoints in the percentages to assign codes (Figure 4.25), and then analyzing by those codes (Table 4.17). No significant differences were found for students in this analysis. Students in both schools performed similarly on this task. The horizontal addition task, which would have been included in this grouping if enough data were available, did show significant differences between the schools, but it is not known if inclusion of this task would have changed the outcome of this analysis. Analysis of this task using only the students who answered it is included at the end of this chapter.

Task	Category	Score
Strips and Squares: Hidden (see Figure 4.12) What level can the student answer addition and missing addend problems.	*0* *1* *2* *3* *4* *5*	0 2 4 6 8 10

Figure 4.24: Hidden task questions, categories, and scoring

0	40	60	80	100	100
0	40	60	80	100	100
0	40	60	80	100	100
0	40	60	80	100	100
0	45	60	80	100	100
0	52.5	70	80	100	100
17.5	52.5	80	80	100	100
17.5	52.5	80	89	100	100
20	60	80	87.5	100	100
35	60	80	87.5	100	100
35	60	80	87.5	100	100
35	60	80	100	100	100
35	60	80	100	100	100
35	60	80	100	100	100
40	60	80	100	100	100
40		80		100	
		low	0-64		
		med	65-93		
		high	94-100		

Figure 4.25: Percentages and break points for mental arithmetic tasks

Table 4.17
Comparison of Montessori and Non-Montessori Student Response Levels on Hidden Strips and Squares Task (N = 93)

	Montessori grade level			Non-Montessori grade level		
Category	1st	2nd	3rd[a]	1st	2nd	3rd[a]
low	6	6	1	12	4	7
med	5	4	4	3	5	1
high	3	7	11	1	7	6

Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 3.38, p<0.20
Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 0.48, p<1.00
[a] Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 7.67, p<0.025
All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 4.89, p<0.10

Flower and bean task group analysis. The second conceptual task grouping contained two tasks: the flower and bean problems. Scores were assigned according to the category of the response from earlier analysis (Figure 4.26). The scores from both tasks were added to find the total score for this section and then divided by the possible score of 20 to find the percent. Percentages ranged from 0 to 100 and were grouped into approximately thirds to find categories for low, medium, and high (Figure 4.27).

Analysis of the categories showed statistically significant differences favoring Montessori students for grade 1 and overall. This result is in accordance with the previous analysis for these tasks.

Both the flower and bean problems require students to think about place value ideas with larger numbers and in more flexible ways. The results show a greater ability of the Montessori students to solve these word problems.

Task	Category	Score
Flower Problem (see Figure 4.13)	*none*	0
How many petals are there in 10 flowers with 10 petals each and two	*10s*	4
more petals?	*100s*	8
Bean Problem (see Figure 4.14)	*none*	0
How many rows of 10 beans can be made with 72 beans?	*10s*	4
How many rows of 10 beans can be made with 123 beans?	*almost 100s*	6
How many cups of 100 beans can be filled with 1,214 beans?	*100s*	8
	almost 1000s	10
	1000s	12

Figure 4.26: Flower and bean task group questions, categories, and scoring

0	15	40	55	80	100
0	15	40	55	80	100
0	15	40	60	80	100
0	15	45	70	80	100
0	15	55	70	90	100
0	15	55	70	90	100
0	15	55	70	90	100
0	15	55	75	90	100
0	30	55	75	100	100
0	40	55	75	100	100
0	40	55	80	100	100
15	40	55	80	100	100
15	40	55	80	100	100
15	40	55	80	100	100
15	40	55	80	100	100
15		55		100	

low	0-50
med	51-84
high	85-100

Figure 4.27: Percentages and break points for flower and bean tasks

Table 4.18
<u>Comparison of Montessori and Non-Montessori Student Response Levels on Flower and Bean Task Group (N = 93)</u>

	Montessori grade level[b]			Non-Montessori grade level[b]		
Category	1st[a]	2nd	3rd	1st[a]	2nd	3rd
low	5	6	1	14	8	1
med	4	4	6	2	6	9
high	5	7	9	0	2	4

[a] Grade 1 Montessori vs. non-Montessori χ^2 (2, n = 30) = 9.84, p<0.01
Grade 2 Montessori vs. non-Montessori χ^2 (2, n = 33) = 3.44, p<0.20
Grade 3 Montessori vs. non-Montessori χ^2 (2, n = 30) = 2.40, p<1.00
[b] All grades Montessori vs. non-Montessori χ^2 (2, N = 93) = 12.07, p<0.01

<u>Conceptual tasks analysis.</u> Since the flower and bean problem group contained two tasks and the hidden strips and squares task was analyzed alone, the score for the conceptual tasks was found by doubling the flower/bean percentage, adding it to the percentage from the hidden task, and then dividing by 1.5. for a total possible score of 200. Scores ranged from 0 to 200 and were grouped into approximately thirds in order to assign a category level of low, medium, or high for each student (Figure 4.28). Analysis of the categories (Table 4.19) show statistically significant differences favoring Montessori students for grade 1 and overall. This result is not surprising because of earlier results.

0	53.3	100	126.7	168.3	200
0	58.3	106.7	126.7	173.3	200
0	60	106.7	133.3	173.3	200
11. 7	73.3	106.7	133.3	173.3	200
20	76.7	106.7	133.3	173.3	200
20	78.3	113.3	140	173.3	200
23.3	80	113.3	140	173.3	200
26. 7	86.7	113.3	146.7	173.3	200
31. 7	86.7	120	146.7	173.3	200
35	86.7	120	146.7	186.7	200
35	86.7	120	160	186.7	200
40	93.3	120	165	186.7	200
43.3	93.3	126.7	166.7	186.7	200
43.3	93.3	126.7	166.7	186.7	200
43.3	100	126.7	166.7	200	200
50		126.7		200	
		low	0-96		
		med	97--162		
		high	163-200		

Figure 4.28: Scores and break points for conceptual tasks

Table 4.19

Comparison of Montessori and Non-Montessori Student Response Levels on Conceptual Tasks (N = 93)

	Montessori grade level[b]			Non-Montessori grade level[b]		
Category	1st[a]	2nd	3rd	1st[a]	2nd	3rd
low	6	5	0	14	4	1
med	2	3	5	2	8	8
high	6	9	11	0	4	5

[a] Grade 1 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 30) = 9.11, $\underline{p}<0.025$

Grade 2 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 33) = 4.28, $\underline{p}<0.20$

Grade 3 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 30) = 3.83, $\underline{p}<0.20$

[b] All grades Montessori vs. non-Montessori χ^2 (2, $\underline{N}$ = 93) = 12.67, $\underline{p}<0.01$

Analysis of mental addition task group using only the subset of students answering the horizontal addition task. The horizontal task question was added after many of the student interviews had been conducted and the results were statistically significant. Therefore, the following analysis was performed to determine if the statistical significance of the conceptual tasks was altered by analyzing only those students who were given the horizontal addition question. Figure 4.29 shows the scoring method for both mental addition tasks (strips and squares hidden task and the horizontal addition task). Previous analysis of the conceptual tasks included only the hidden task question.

A total of 19 points was possible on the mental addition task group and percentages were found for each of the 54 students in the subset (Figure 4.30). Students scores were then converted to the high, medium, low scoring and were analyzed using Chi squared techniques (Table 4.20). Results show statistical significance favoring Montessori students for grades 1 and 3 and overall.

Task	Categories	Score
Strips and Squares: Hidden (see Figure 4.11) What level can the student answer addition and missing addend problems.	*0* *1* *2* *3* *4* *5*	0 2 4 6 8 10
Horizontal Addition Task (see Figure 4.12) How does the student solve horizontal addition questions mentally.	*none* *1s* *10s/1s* *algorithm*	0 3 6 9

Figure 4.29: Mental addition group questions, categories, and scoring using only the subset of students answering the horizontal addition task

0	52	89
9	57	89
15	57	89
15	59	89
18	61	100
25	63	100
26	63	100
31	68	100
31	73	100
34	78	100
36	78	100
42	84	100
42	84	100
43	84	100
47	84	100
52	84	100
52	84	100
52	84	100

low 0-54
med 55-86
high 87-100

Figure 4.30: Percentages and break points for mental addition group tasks using only the subset of students answering horizontal addition task

Table 4.20

Comparison of Montessori and Non-Montessori Student Response Levels on the Mental Addition Task Grouping Using Only the Subset of Students Answering the Horizontal Addition Task (n = 54)

	Montessori grade level[c]			Non-Montessori grade level[c]		
Category	1st[a]	2nd	3rd[b]	1st[a]	2nd	3rd[b]
low	3	2	1	10	1	2
med	2	1	1	2	4	7
high	4	6	5	0	3	0

[a] Grade 1 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 21) = 7.49, $\underline{p}<0.025$

Grade 2 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 17) = 3.09, $\underline{p}<1.00$

[b] Grade 3 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 16) = 9.74, $\underline{p}<0.01$

[c] All grades Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 54) = 15.13, $\underline{p}<0.001$

Analysis of conceptual tasks using only the subset of students answering the horizontal addition task. Analysis of the responses of the subset of 54 students answering the horizontal addition question proceeded by adding the percentages from the mental addition tasks and the flower and bean tasks. These scores and the break points for assigning codes of low, medium, and high are shown in Figure 4.31. Chi Squared analysis of the categories showed statistical significance favoring Montessori students in grades 1 and 3 and overall.

0	107	173
9	107	175
15	112	175
15	112	180
33	115	184
34	118	184
36	129	189
40	132	189
43	133	190
46	139	200
46	141	200
52	144	200
81	144	200
97	148	200
102	159	200
103	159	200
103	164	200
107	164	200

low 0-105
med 106-168
high 169-200

Figure 4.31: Scores and break points for conceptual tasks using only the subset of students answering horizontal addition task

Table 4.21

Comparison of Montessori and Non-Montessori Student Response Levels on Conceptual Tasks Using Only the Subset of Students Answering the Horizontal Addition Task (n = 54)

	Montessori grade level[c]			Non-Montessori grade level[c]		
Category	1st[a]	2nd	3rd[b]	1st[a]	2nd	3rd[b]
low	1	2	0	10	2	2
med	4	2	2	2	4	6
high	4	5	5	0	2	1

[a] Grade 1 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 21) = 11.84, $\underline{p}<0.01$

Grade 2 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 17) = 1.90, $\underline{p}<1.00$

[b] Grade 3 Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 16) = 6.52, $\underline{p}<0.05$

[c] All grades Montessori vs. non-Montessori χ^2 (2, $\underline{n}$ = 54) = 14.82, $\underline{p}<0.001$

Summary. The final levels of analysis confirmed earlier results: that when taken as a whole, the Montessori students consistently outperformed the non-Montessori students on tasks of a more conceptual nature, while performing the same or slightly better on more procedural tasks. Whether the horizontal addition task was included or excluded, Montessori students significantly responded at higher levels than the non-Montessori students on conceptual tasks in grade 1 and overall. However, when the horizontal addition task was included, the subset of Montessori students outperformed their counterparts on conceptual tasks in grades 1 and 3 and overall. Graphs of conceptual and procedural scores for students are shown in Appendix D.

CHAPTER 5
SUMMARY AND IMPLICATIONS

This dissertation began by stating that poor mathematics achievement of U. S. students has been a concern to educators and the public. Research has shown that this may be because students' understanding of mathematics—and specifically for this study, place value concepts—are hindered by the mostly procedural treatment of mathematics in classrooms. The purpose of this study was to find out if students from a Montessori school and a non-Montessori school had different skills and understanding of place value concepts. Nearly 50 students from each of two schools were interviewed to establish responses to a broad range of tasks involving place value. Data revealed statistically significant differences in student responses favoring students from the Montessori school on tasks of a more conceptual nature, while responses to tasks of a more procedural nature were generally similar for all students.

Implications

Since this research was designed to be a descriptive study and not experimental, correlations between more advanced responses and type of school cannot be construed as causal relationships. Many other factors may be at work. Therefore, the implications listed below must remain tentative, in anticipation of further research.

Implications for Theory

Dienes' theory. The Dienes framework calls for a dynamic and constructive learning environment involving perceptual variability (multiple embodiments) and mathematical variability. The Montessori school in this study provided this type of environment for students: one that allowed children the freedom to use many materials (perceptual variability) individually (dynamic) in order to formulate their own conceptions (constructive) of the ideas of tens, hundreds, and thousands (mathematical variability). Meanwhile, the environment in the non-Montessori classrooms included direct instruction with limited use of manipulative materials while teaching place value information in a piecemeal fashion.

While it is not certain from this study which of the four features may have the most influence on the Montessori students' learning, together the features seemed quite powerful for students' concepts of place value. However, the feature the author would like to highlight because it is the most overlooked aspect, even in reform classrooms, is the mathematical variability principle. As mentioned previously, students in the non-Montessori classrooms, as is typical for many elementary mathematics curricula, were not exposed to large numbers until the second or third grades, while students in the Montessori school learned about these numbers in preschool and first grade. If students learn first about tens, then hundreds, and then thousands in isolation of each other, with hundreds and thousands work being treated mostly symbolically, conceptual links have little chance of forming. These students cannot begin to formulate their ideas of the structure of the place value system, and instead may see each round of larger numbers as a new topic for study. The Montessori curriculum, because it includes manipulative materials and activities involving tens, hundreds, and thousands simultaneously, may allow these students to consider the deeper structure of place value ideas instead of just surface features. This may be why the Montessori students in this study had more flexibility in thinking about tasks involving conceptions of place value into the hundreds and thousands.

Therefore, the use of manipulative materials that are meant to embody multiple representations of concepts, the Montessori teachers' role in providing an environment which is constructive and dynamic, and the treatment of large numbers in the Montessori school may have helped to increase students' ability to generalize place value concepts. This lends credibility to Dienes' dynamic, constructive, mathematical variability, and perceptual variability (multiple embodiments) principles as a way to increase a student's ability to generalize concepts.

Jones et al. framework. This study used many tasks that encompass the place value framework created by Jones and Thornton (1993; Jones, Thornton, & Putt, 1994; Jones, Thornton, Putt, Hill, Mogill, Rich, & Van Zoest, 1996). This framework was created by analyzing and interpreting results from children in a socioconstructivist environment in which learning evolved from the posing of verbal tasks to children after which the children shared and discussed their solutions. The children in the current study were from different learning environments than that described in the Jones and Thornton studies. However, the questions involved in the current study were not posed in the same hierarchical manner as in the Jones and Thornton studies. Therefore, this author can only comment on the components of the framework and not the hierarchical nature of its proposed tasks.

The tasks involving two of the four components, counting and number relationships, showed little or no differences between the Montessori and non-Montessori students in this study. However, the tasks of other components, grouping and partitioning, showed significant differences favoring Montessori students. Therefore, it can be assumed that different learning environments affect the different components of place value differently. More research using the Jones and Thornton framework should be undertaken to determine if the levels of the framework are stable in learning environments other than those used in their studies.

Foreign language number learning. One group of studies cited in the literature review dealt with first graders speaking an Asian language (Miura, Okamoto, Chungsoon, Steere, & Fayol, 1994), in which spoken number words parallel the structure of the base-ten number system, as compared to students speaking other languages, which do not have base-ten features. Since the Montessori students in this study were exposed to numbers in the hundreds and thousands as kindergartners, which is near the time the Asian-language students in the cited studies were learning the words for these numbers, they may have been given a similar advantage to the Asian-language students. This study may provide more evidence for the theories of these authors.

Implications for Teaching

Use of manipulatives. Some teachers express concern with the use of manipulative materials as a crutch that will hinder students' development of procedures in mathematics, just as many teachers and the public lament the use of calculators or adding machines as a crutch to thinking. However, data from this study showed that students who had used manipulatives for extended periods of time (Montessori students) were able to conceptualize tasks and provide higher-level responses using more generalized concepts of place value. In the constructive atmosphere of the Montessori classroom, it seems that the scaffolding provided by the teacher (both verbally and in the form of offering, creating, and adapting materials) and the materials in the child-centered environment allowed the students to use manipulatives as a tool instead of a crutch. The Montessori students' abilities to outperform the non-Montessori students on conceptual tasks while scoring relatively the same on more symbolic tasks shows that Montessori students have not been hindered on either conceptual or procedural tasks by using the materials in their classrooms.

In addition, the teacher's goal in the Montessori classrooms was the internalization of symbolic mathematics, not just concepts and manipulation of materials, but the move to symbolism was provided only when the child was ready. Therefore, if teachers want to provide a constructive environment for children in which they learn both concepts and procedures, the type, hierarchy, and consistency of manipulative materials should be considered, keeping in mind Dienes' mathematical variability principle to provide work with materials involving deeper structures of mathematics.

Type of instruction. Another factor that may play a part in the Montessori students' achievement is the use of small group and individualized instruction. Therefore, these methods of teaching should be considered whenever possible.

Assessment. A final implication for teaching is the use of testing. Students in both schools performed similarly on items commonly used as assessments for place value (e.g., counting, using the standard addition algorithm for multidigit numbers), but showed differences on other items requiring more conceptual understanding of the ideas. Therefore, classroom assessment should be ongoing and include tasks involving all aspects of place value (counting, grouping, partitioning, and number relationships in addition to symbolic treatments) in order to get a better picture of students' whole understanding. Otherwise, teachers may be fooled into thinking that students are ready to move on to a more abstract treatment of a topic before they have a concrete understanding.

Limitations of the Study

Limitations of the study include the inability of the author to more precisely describe the home backgrounds of the students in this study. Socioeconomic status is a major indicator of academic performance, and without this information, one cannot be sure that students in one school had an economic, and therefore an academic, advantage over other students.

Another limitation is the lack of a complete description of the context of students' learning. A complete and in-depth study of the two different learning environments may lead to further insights into the learning of the students.

The descriptive nature of the study, as previously discussed, also provided limitations for the conclusions and implications drawn. Hence, further study is needed to ascertain more generalizable results.

Further Study

This study has provided the author with many more questions than answers. The main questions are: How can educators best teach mathematics to students so they learn both conceptual and procedural understanding of topics? And, How can teacher educators foster this kind of teaching?

Further study of Montessori school student understanding could include various replications of this study to include other topics, comparisons with other types of schools and philosophies, and expansion to include a wider-scale study to include many more students. Some questions for study of students' learning are:

- How would Montessori students fare on other topics in the elementary curriculum, including, but not limited to, decimals, multiplication and division, fractions, and problem solving?
- What would be the differences and similarities in abilities of students in a Cognitively Guided Instruction classroom vs. students in a Montessori school, since both contexts focus on children's thinking and may use small-group instruction?
- What would be the result of replicating this study comparing public Montessori and public non-Montessori schools, since the students in these two types of schools may have more similar socioeconomic status and educational backgrounds?
- What would be the result of replicating this study with larger numbers of students chosen randomly from Montessori schools and non-Montessori schools in an area?
- What gender differences in mathematics learning are found in Montessori students?

Studies of the environment of Montessori classrooms and of Montessori teachers may also shed light upon what types of teaching and philosophy help to foster the mathematics understanding found in Montessori students. A full comparison of the contexts of learning of both groups of students in this study may shed more light upon their differences and highlight which features may be good subjects for in-depth study. Changes made to a Montessori classroom may be studied to investigate differences in children's learning. Some questions for study of the learning environment are:

- What similarities and differences would be found from an in-depth study of the two learning environments found in this current study?
- In the interviews with the teachers in the two schools, the Montessori teachers spent much of the time focusing their responses on student learning and the non-Montessori

teachers focused their responses on the curriculum. To what do the Montessori teachers attribute their beliefs and teaching style: type of person they are, classroom and school atmosphere of a Montessori school, or their Montessori training?

- What would be the outcomes if a large problem-solving component (teaching through the posing of word problems and situations) were added to current Montessori practice?
- What similarities or differences would be found in an investigation of the teachers' conceptions of the sequence of place value ideas? Do teachers see a connection between ideas of counting, place value, and arithmetic operations?

Finally, study of Montessori student conceptions and the learning environments that promote such mathematical understanding can be used in inservice and preservice teacher education and projects to help promote mathematical understanding in other environments. Some questions for the study of teacher education include:

- How could pieces of the Montessori curriculum, materials, and methods of teaching be sequenced to fit into a more traditional atmosphere?
- What philosophies would help preservice and inservice teachers to view mathematics through their students' eyes?
- What type and sequence of materials may be helpful and/or necessary to foster students' understanding of place value and other topics?

Conclusions

This study used a modified version of a framework originally developed by Dienes. Extension of the framework to place value ideas included tasks to assess the understanding of numbers larger than are usually discussed in the literature. This was done since the Dienes framework calls for more and more abstract generalizations, and for place value, this means larger and larger numbers. For instance, the flower and bean tasks both attempt to see how developed are a child's ideas of place value into the hundreds and thousands, instead of just the tens. The Montessori school curriculum provides work with these types of large numbers in the thousands for kindergartners and first graders, which is not done until much later in the non-Montessori school. Therefore, it might be assumed that by the third grade, students at the non-Montessori school would have *caught up* to the Montessori students in their responses to tasks involving place value into the thousands.

However, the thousands work provided the students in the later grades at the non-Montessori school consisted mostly of lecture and workbook practice which was mostly of a procedural nature. The evidence in this study shows that the students from both schools responded similarly on procedural tasks but differently on conceptual tasks. By the third grade however, these differences mostly disappear except for a few tasks. When asked to perform horizontal addition problems, non-Montessori third graders almost exclusively added by using the standard addition algorithm even though it is awkward and many mistakes were made. The Montessori students, even though they could perform the algorithm, instead chose a faster and more accurate method of adding using tens and ones, showing more flexible thinking. Therefore, although their responses on many tasks traditionally used as assessment for place value concepts were similar, these students do seem to be different. A definitive reason for this difference will have to wait until further study.

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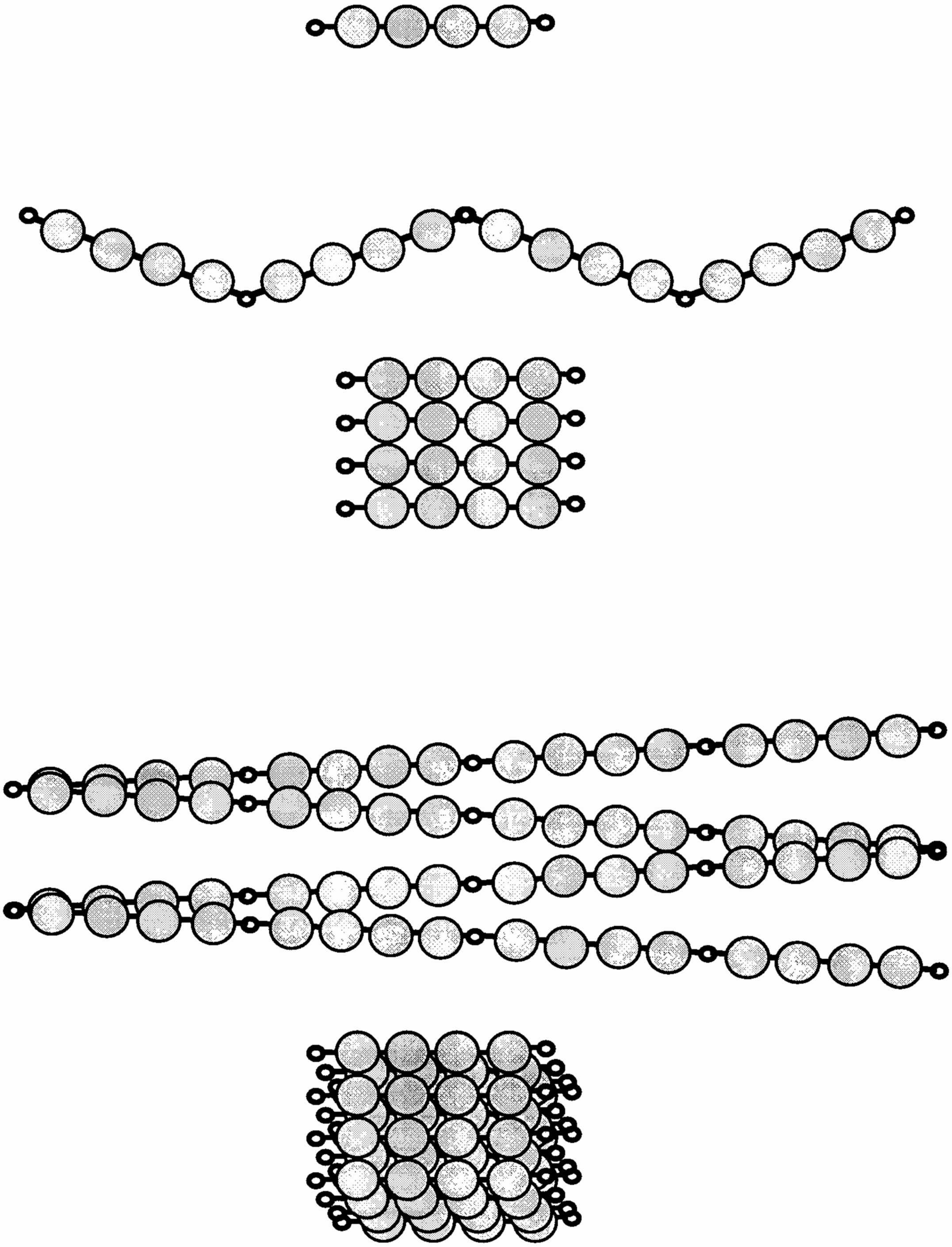
APPENDIX A

Montessori Yellow Bead Bar Set

Montessori Yellow Bead Bar Set



APPENDIX B

Oral Script to Student

Oral Script to Student

I'd like to talk with you for about 30 minutes today about math. I want to ask you some questions that you may or may not have seen in your classroom. This interview won't affect your grades and your teacher and parents won't know how you answered the questions. Your answers are only to help me to understand how kids think about math.

I want to videotape the interview, if that's OK with you. When I write up my report, I won't use your real name. You don't have to answer any questions that you don't want to, and you can ask to stop the interview at any time. Would you be willing to participate?

APPENDIX C

Interview Protocol

Interview Protocol

General questions

How long have you been here at this school?

Where did you go to kindergarten?

What do you do in math class?

What materials do you use in math class?

Counting task

How high can you count?

Would you start at 35 and count?

Would you count by 10s?

Would you start at 60 and count by 10s?

Would you count by 100s?

Would you start at 700 and count forward by 100s?

Would you start at 24 and count backwards?

Would you start at 60 and count backwards by 10s?

Unifix cubes task

Would you guess how many are here?

Would you count all these cubes for me?

How could you put them so the next child could count them even faster?

Strips and squares tasks

Counting task

Would you put 13 squares on this paper?

Would you put 86 squares on this paper?

If I take off this (one strip) how many left?

(If child did not use a "ten" for the 13) Would you put 18 squares on this paper?

Uncovering task

Would you tell me how many squares are here? (each time another group is uncovered

Hidden squares/strips tasks

(2 strips and 3 squares visible) "I've hidden 4 squares and 1 strip; how many are there all together?"

(3 strips and 8 squares visible) "I've hidden 4 squares and 1 strip; how many are there all together?"

(3 strips visible) "I've hidden some under here; If there are 50 altogether, how many are covered?

(2 strips and 4 squares visible) "If there are 36 in all, how many are covered?"

(2 strips and 3 squares visible) "If there are 41 in all, how many are covered?"

Horizontal addition task

16+9=

28+13=

38+24=

39+53=

Flower task

"What if you went outside to pick some flowers for (principal's name). Each flower has 10 petals. You found 2 loose petals on the ground and picked 10 whole flowers. How many petals would you have?"

If child is unable to answer, she is asked about 3 loose petals and 4 whole flowers.

Bean task

(put 10 beans in a row on table) "If you had beans like these, and you put beans in rows of 10, how many full rows of 10 beans could you make from 35 beans?" "from 72 beans?" "from 123 beans?"

"If you put 100 beans in a cup, how many full cups could you fill from 423 beans?" "from 1,214 beans?"

Symbolic tasks

Number cards task

Show card with 1,247 and 1,326. "Which is bigger?" "Why?"

Show card with 358 and 372. "Which is bigger?" "Why?"

Digit cards task

Show cards with digits written on them: 1, 3, 4, 5. Show that 1 next to the 3 makes a 13.

"What is the biggest number you can make with these cards? Can you make an even bigger number?"

Vertical addition task

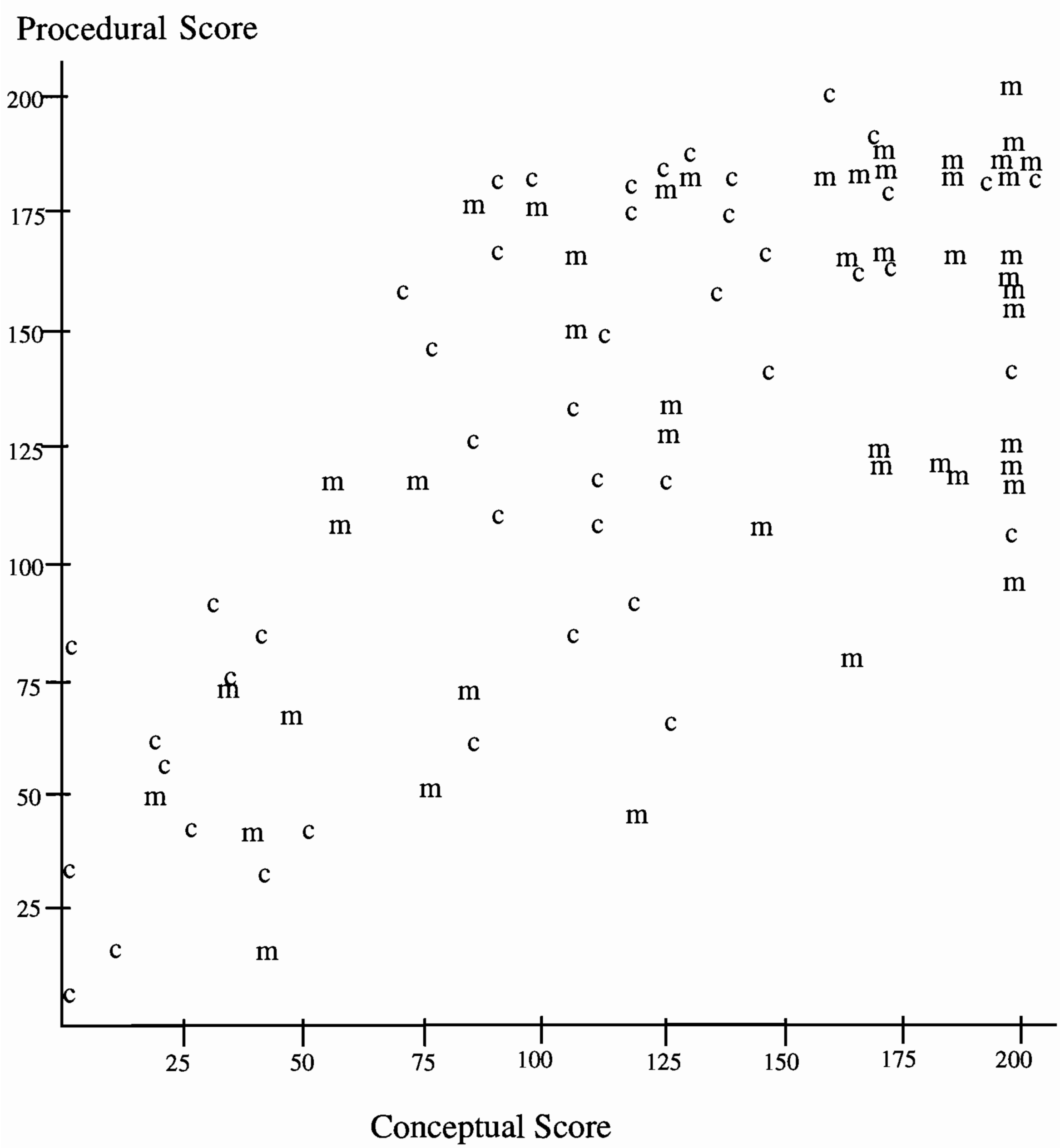
Student is asked which problems he can do and is then requested to pick the hardest on their list and solve. Students are permitted to use a pencil.

13 + 48; 125 + 492; 2,474 + 5,823; 68,183 + 27,207; 134,223 + 245,912

APPENDIX D

Graphs of Conceptual and Procedural Scores

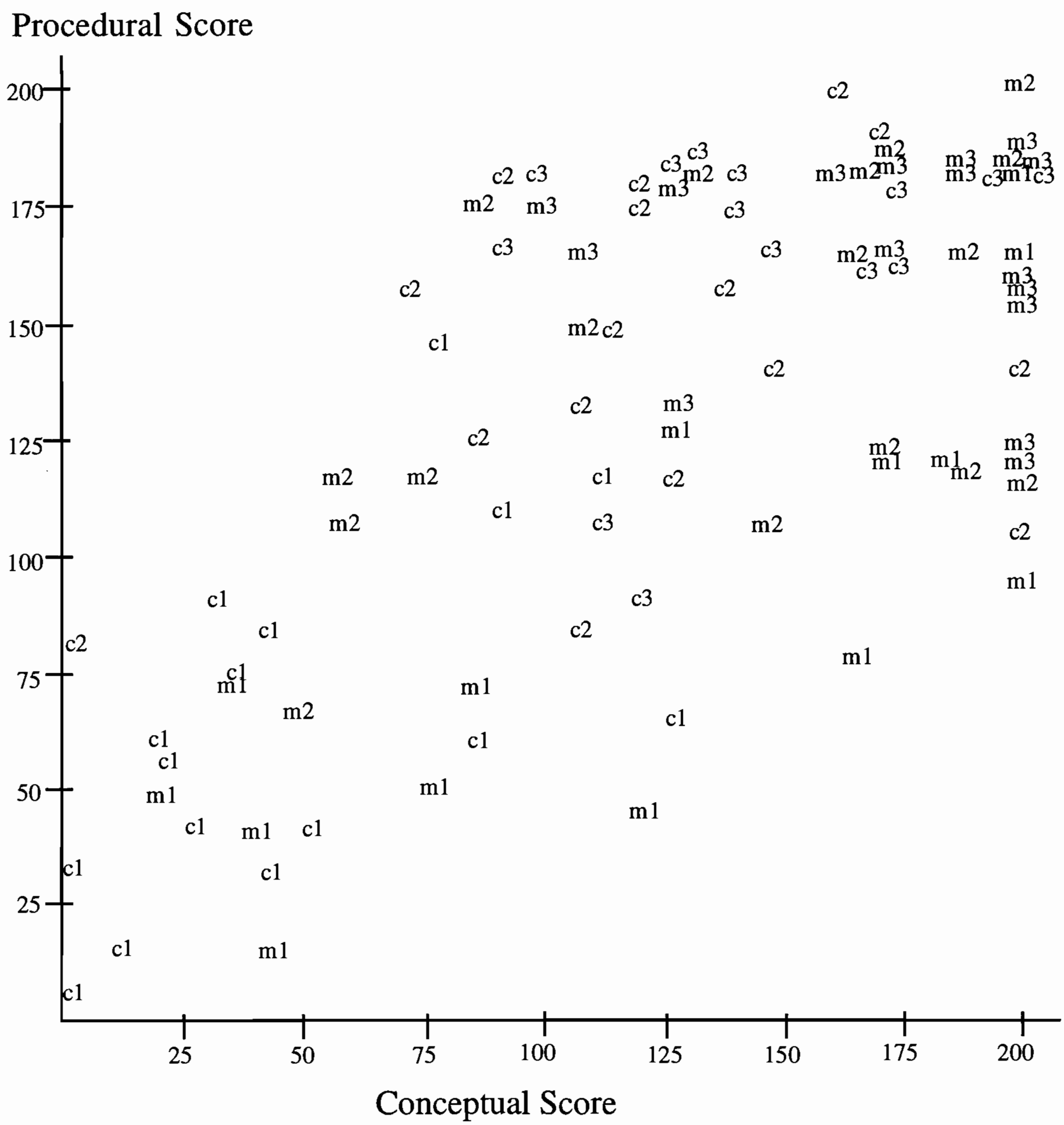
Graph of Conceptual and Procedural Scores of Montessori vs. Non-Montessori Students



m Montessori students
c Comparison (non-Montessori) students

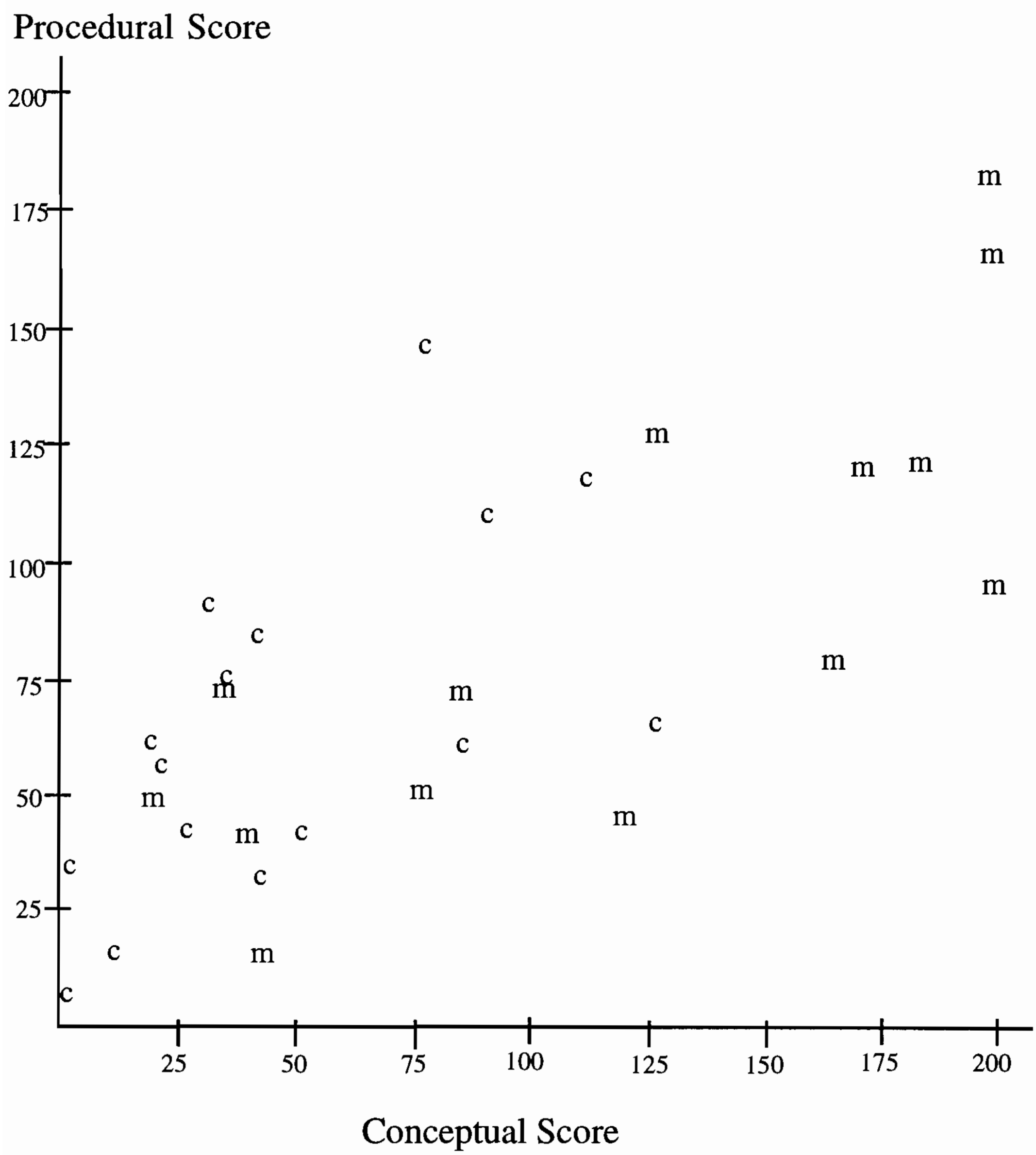
Graph of Conceptual and Procedural Scores of Montessori vs. Non-Montessori

Grades 1, 2, and 3 Students



m1	Montessori grade 1 students	c1	Comparison grade 1 students
m2	Montessori grade 2 students	c2	Comparison grade 2 students
m3	Montessori grade 3 students	c3	Comparison grade 3 students

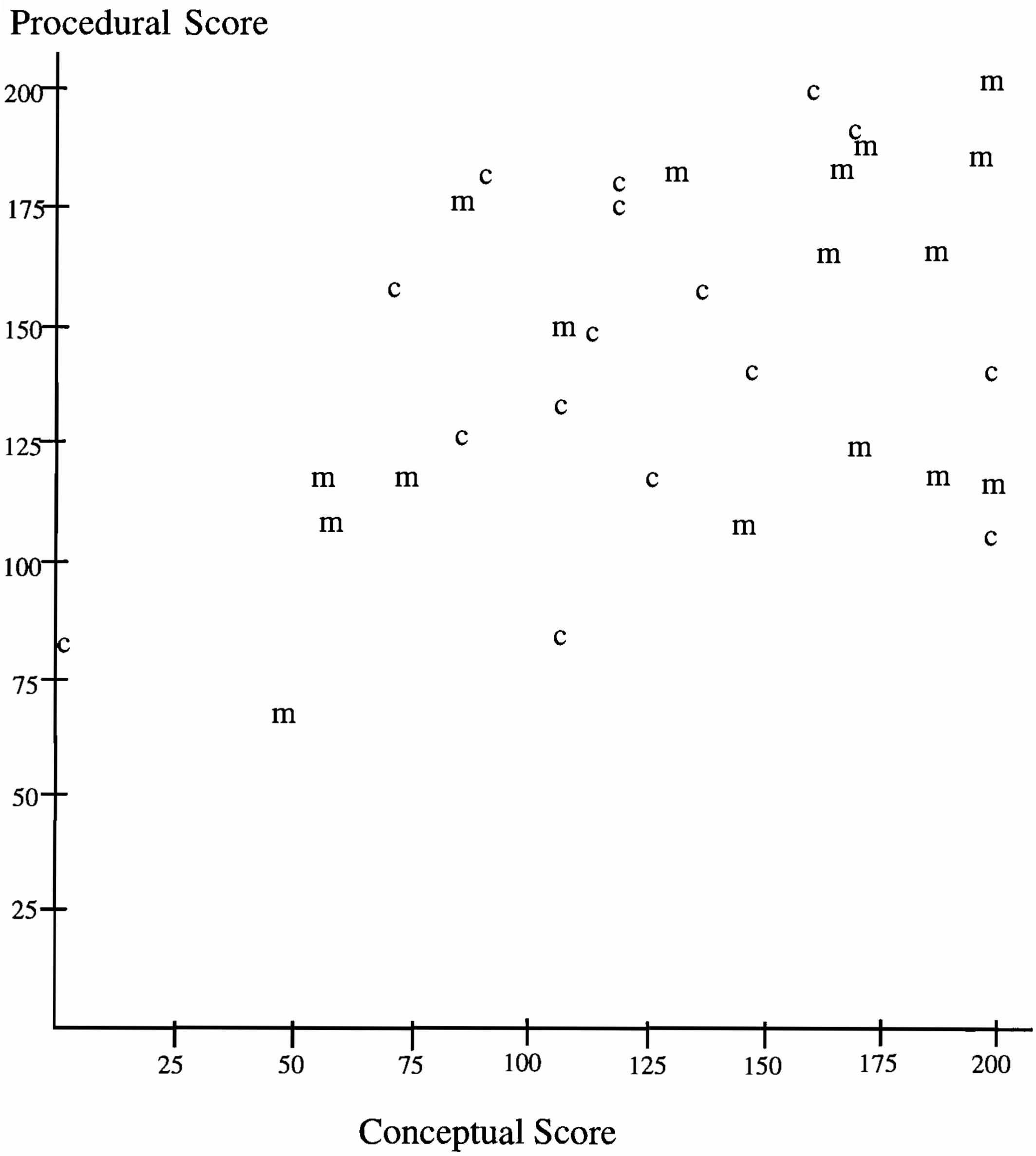
Graph of Conceptual and Procedural Scores of Montessori vs. Non-Montessori Grade 1 Students



m Montessori grade 1 students
c Comparison grade 1 students

Graph of Conceptual and Procedural Scores of Montessori vs. Non-Montessori

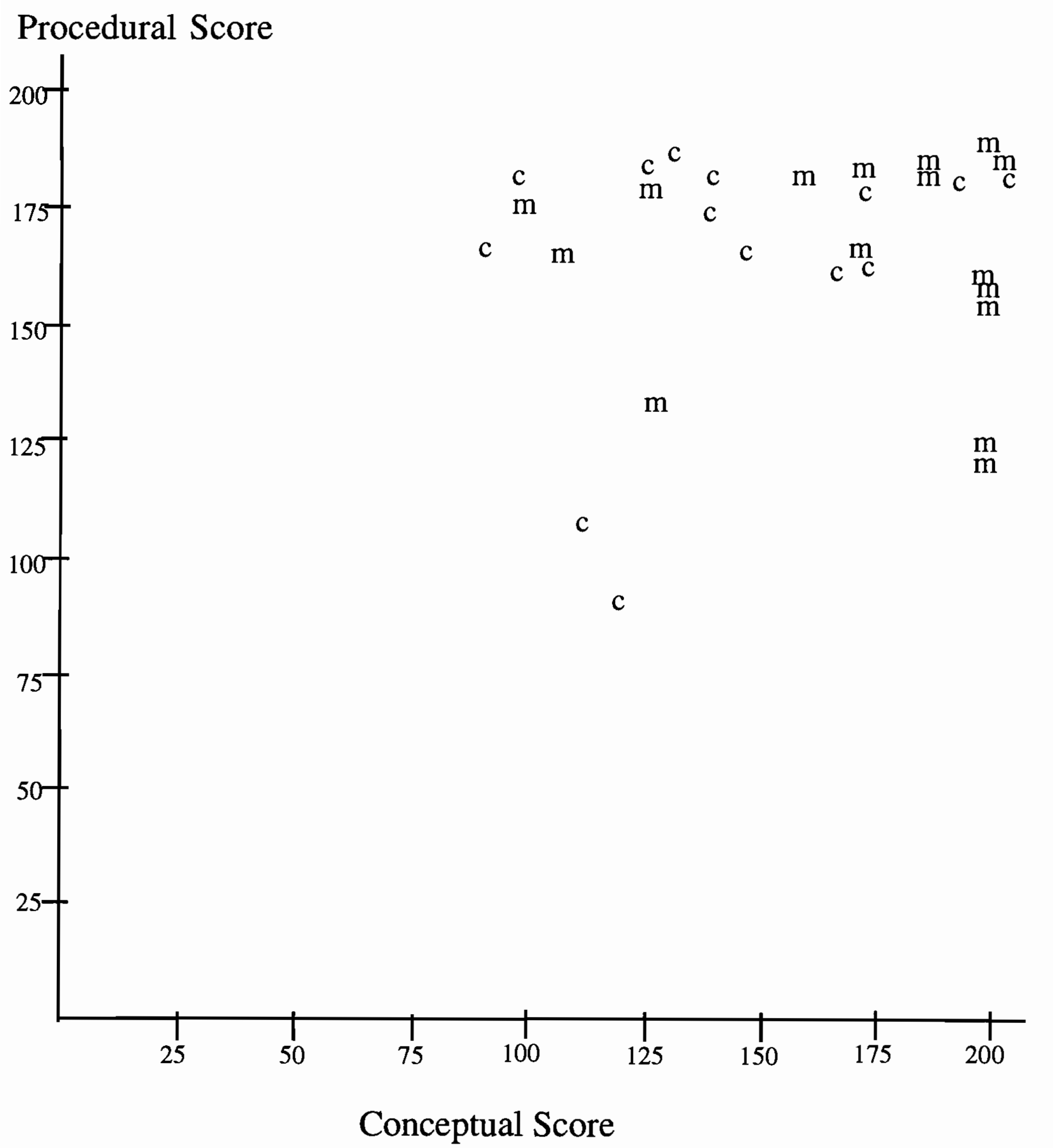
Grade 2 Students



m Montessori grade 2 students
c Comparison grade 2 students

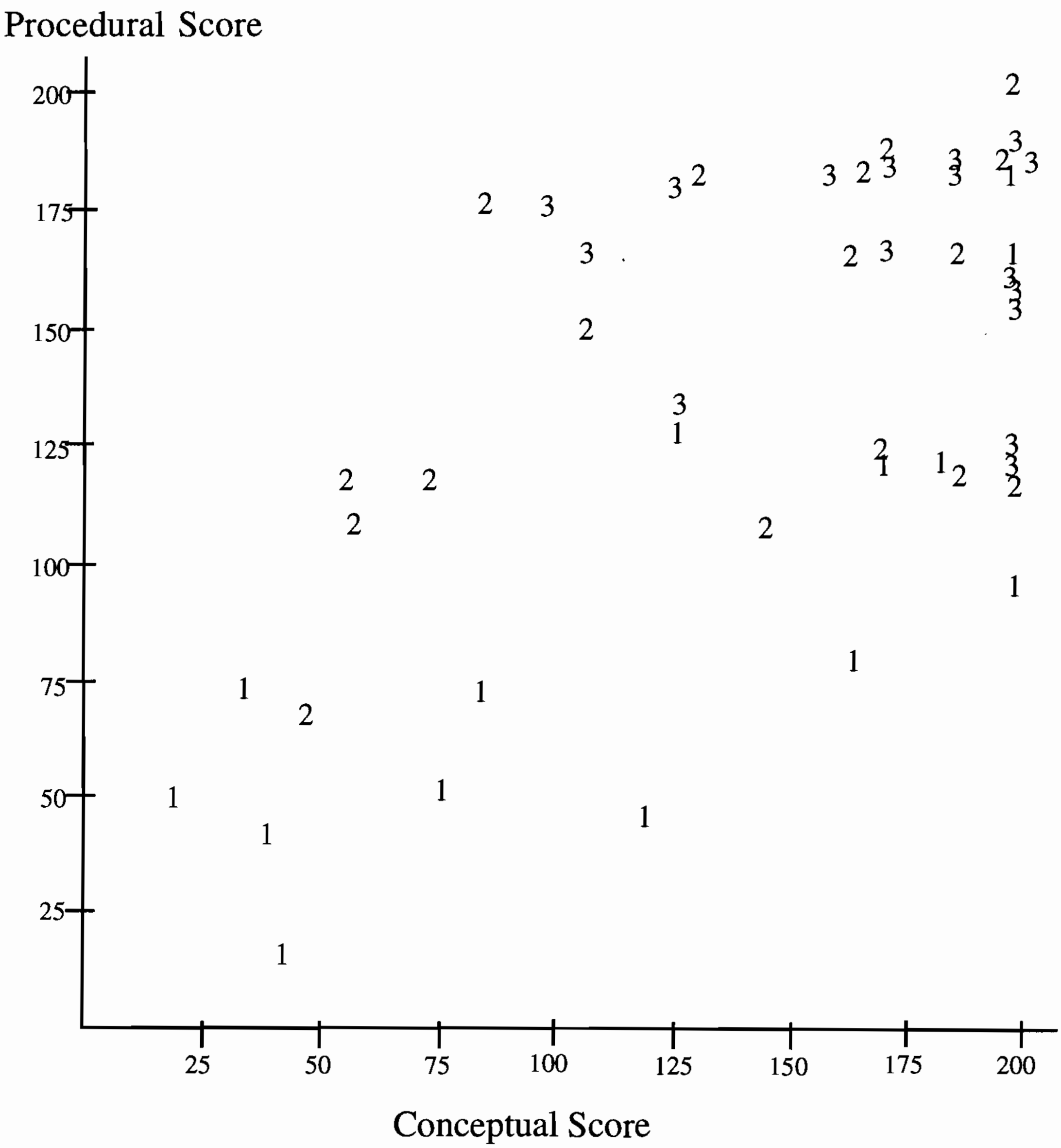
Graph of Conceptual and Procedural Scores of Montessori vs. Non-Montessori

Grade 3 Students



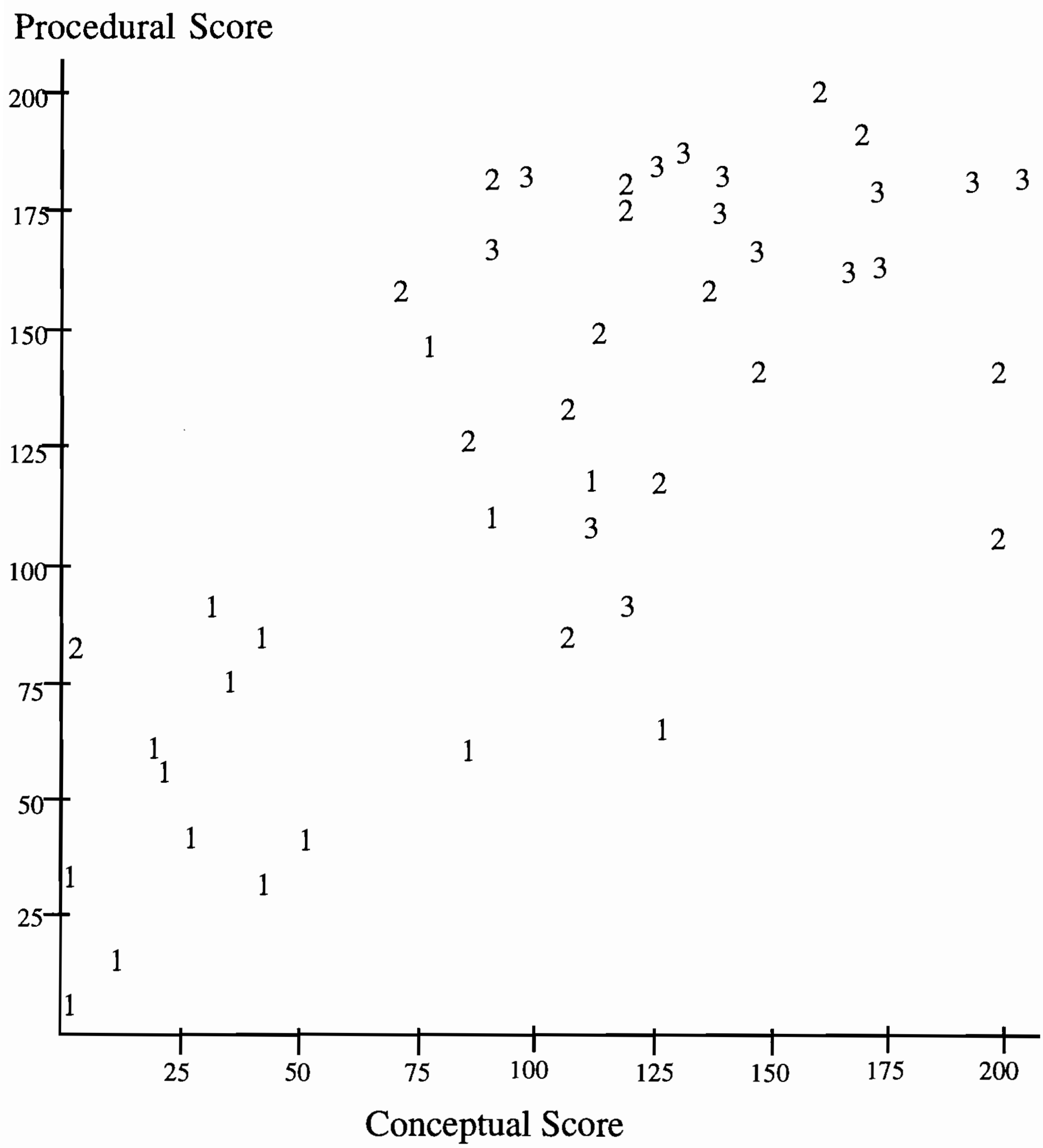
m Montessori grade 3 students
c Comparison grade 3 students

Graph of Conceptual and Procedural Scores of Montessori Grade 1, Grade 2, and Grade 3 Students



1 Montessori grade 1 students
2 Montessori grade 2 students
3 Montessori grade 3 students

Graph of Conceptual and Procedural Scores of Comparison Grade 1, 2, and 3 Students



1 Comparison grade 1 students
2 Comparison grade 2 students
3 Comparison grade 3 students